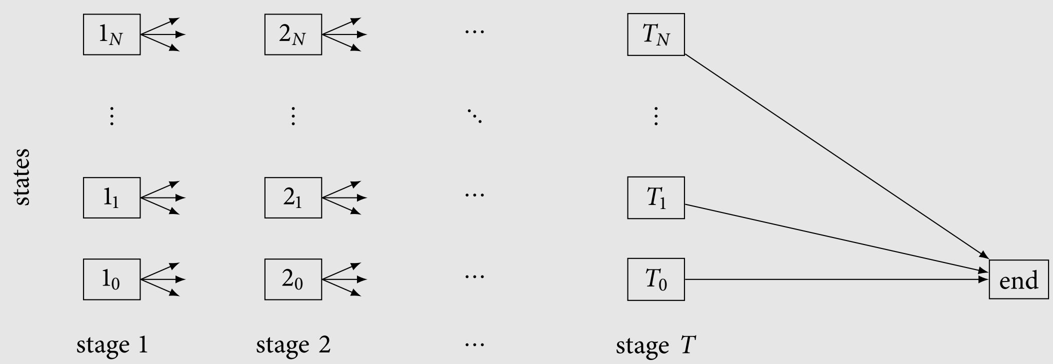
Summary of Models

The shortest path problem

- Data:
  - Directed graph (N, E)
  - **Source node** $s \in N$ and **target node** $t \in N$ $(s \neq t)$
  - Each edge (i, j) in E has a **length** c_{ij}
- The **length of a path** is the sum of the lengths of the edges in the path
- Problem: Find a path from s to t with the shortest length

Dynamic program – shortest/longest path representation

- **Stages** $t = 1, 2, \dots, T$ and **states** $n = 0, 1, 2, \dots, N$
- Directed graph:



  - Node $t_n \longleftrightarrow$ being in state n at stage t
    - Nodes for the tth stage are put in the tth column
  - Edge $(t_n, (t+1)_m) \longleftrightarrow$ the **decision** to go to state m from state n at stage t
    - Length of this edge = **contribution** (i.e. cost or reward) of making this decision
    - An edge must connect a node in the tth column to a node in the $(t+1)$st column
  - All nodes for the last stage are connected to an "end" node
    - These edges often have a length 0, but not always
- Shortest/longest path problem:
  - Source node = one of the first stage nodes (depends on the problem)
  - Target node = end node

If the contributions (edge lengths) are ...	Then find the ...
rewards	longest path from source to target
costs	shortest path from source to target

Dynamic program – recursive representation

- **Stages** $t = 1, 2, \ldots, T$ and **states** $n = 0, 1, 2, \ldots, N$
- Allowable **decisions** x_t at stage t and state n for $t = 1, \ldots, T-1$ and $n = 0, 1, \ldots, N$
- **Contribution** (i.e. cost or reward) of decision x_t at stage t and state n for $t = 1, \ldots, T$ and $n = 0, 1, \ldots, N$
- **Value-to-go** function $f_t(n)$ at stage t and state n (i.e. cost-to-go or reward-to-go) for $t = 1, \ldots, T$ and $n = 0, 1, \ldots, N$
- **Boundary conditions** on $f_T(n)$ at state n for $n = 0, 1, \ldots, N$
- **Recursion** on $f_t(n)$ at stage t and state n for $t = 1, \ldots, T-1$ and $n = 0, 1, \ldots, N$

$$f_t(n) = \min_{x_t \text{ allowable}} \text{or} \max \left\{ \begin{pmatrix} \text{contribution of} \\ \text{decision } x_t \end{pmatrix} + f_{t+1}\begin{pmatrix} \text{new state} \\ \text{resulting} \\ \text{from } x_t \end{pmatrix} \right\}$$

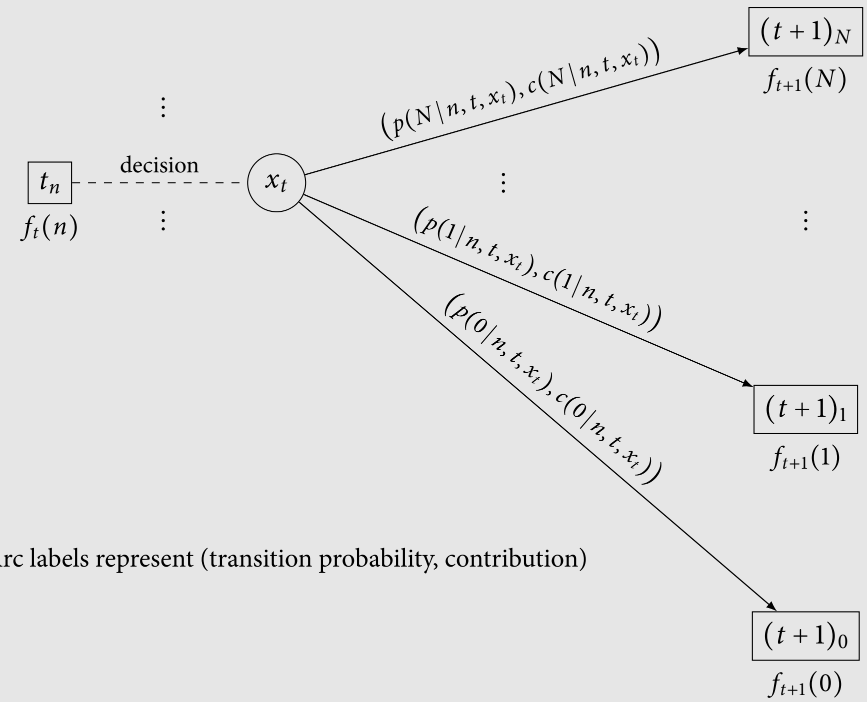
- **Desired value-to-go function value**, usually $f_1(m)$ for some state m

Dynamic program – correspondence between shortest/longest path and recursive representations

Shortest/longest path		Recursive
node t_n	$\leftrightarrow$	state n at stage t
edge $(t_n, (t+1)_m)$	$\leftrightarrow$	allowable decision x_t in state n at stage t that results in being in state m at stage $t+1$
length of edge $(t_n, (t+1)_m)$	$\leftrightarrow$	contribution (i.e. cost or reward) of decision x_t in state n at stage t that results in being in state m at stage $t+1$
length of shortest/longest path from node t_n to end node	$\leftrightarrow$	value-to-go function $f_t(n)$ (i.e. cost-to-go or reward-to-go)
length of edges (T_n, end)	$\leftrightarrow$	boundary conditions $f_T(n)$
shortest or longest path	$\leftrightarrow$	recursion is min or max: $f_t(n) = \min_{x_t \text{ allowable}} \text{or} \max \left\{ \begin{pmatrix} \text{contribution of} \\ \text{decision } x_t \end{pmatrix} + f_{t+1}\begin{pmatrix} \text{new state} \\ \text{resulting} \\ \text{from } x_t \end{pmatrix} \right\}$
source node 1_n	$\leftrightarrow$	desired value-to-go function value $f_1(n)$

Stochastic dynamic program

- **Stages** $t = 1, 2, \dots, T$ and **states** $n = 0, 1, 2, \dots, N$
- Allowable **decisions** x_t at each stage t and state n
- **Transition probability** $p(m \mid n, t, x_t)$ of moving from state n to state m in stage t under decision x_t
- **Contribution** $c(m \mid n, t, x_t)$ for moving from state n to state m in stage t under decision x_t



- **Value-to-go** function $f_t(n)$ at each stage t and state n
- **Boundary conditions** on $f_T(n)$ for each state n
- **Recursion** on $f_t(n)$ at stage t and state n

$$f_t(n) = \min_{x_t \text{ allowable}} \left\{ \sum_{m=0}^{N} p(m \mid n, t, x_t) \Big[c(m \mid n, t, x_t) + f_{t+1}(m) \Big] \right\}$$

for $t = 1, 2, \dots, T-1$ and $n = 0, 1, \dots, N$

- **Desired value-to-go**, usually $f_1(m)$ for some state m