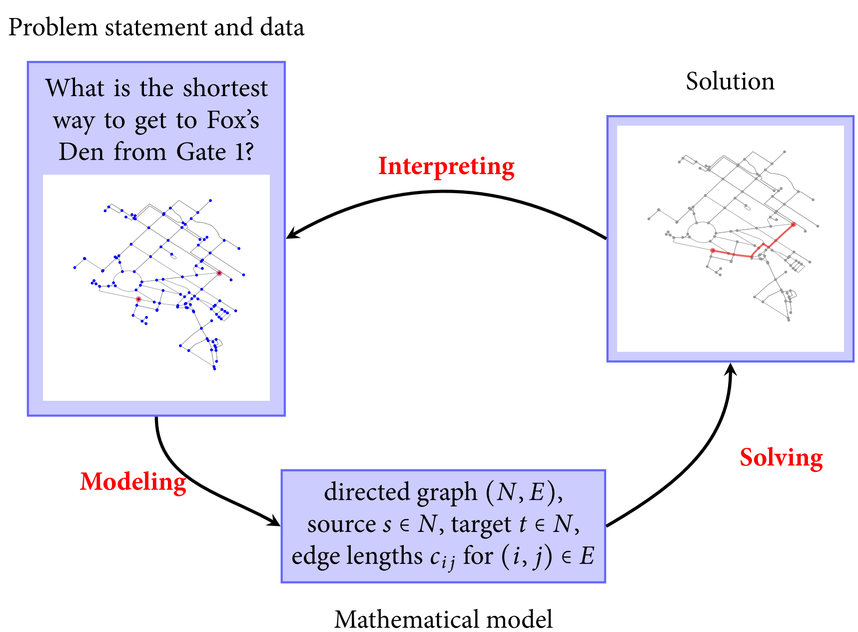
Lesson 1.

Introduction, The Shortest Path Problem

1 Goals for this course

- A course in **operations research**: the discipline of applying advanced mathematical methods to help make better decisions
- Formulate **mathematical models** for real-world **decision-making** problems:
  - The shortest path problem
  - Dynamic programming
  - Markov decision processes
- Use **computational tools** to solve these models with medium-to-large scale data
  - Python and its many data science packages (e.g. `pandas`, `networkx`)
  - Focus on
    - setting up models with the help of design patterns
    - analyzing and interpreting solutions
- Analyze and interpret solutions to these models



- Detailed topic list and schedule on the syllabus

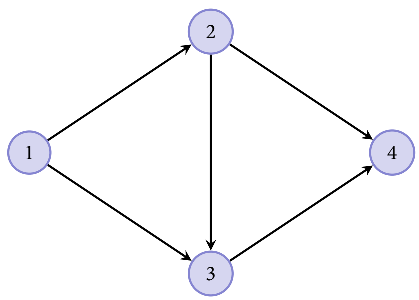
2 This lesson...

- What is the shortest way to get from Point A to Point B?

3 Graphs and networks

- **Graphs** model how various entities are connected
- A **directed graph** (also known as a **digraph**) (N, E) consists of
  - set of **nodes** N (also known as **vertices**)
  - set of **edges** E (also known as **arcs**)
    - arcs are directed from one vertex to another
    - arc from vertex i to vertex j is denoted by (i, j)

Example 1.



4 Graphs are everywhere

- Physical networks – e.g. road networks
- Abstract networks – e.g. organizational charts
- Others?

5 Paths

- A **path** is a sequence of edges connecting two specified nodes in a graph:
  - Each edge must have exactly one node in common with its predecessor in the sequence
  - Edges must be passed in the forward direction
  - No node may be visited more than once

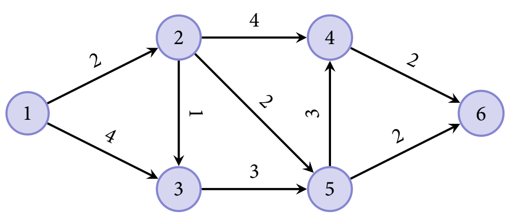
Example 2. Give some examples of paths from node 1 to node 4 in the network in Example 1.

6 The shortest path problem

The shortest path problem

- Data:
  - Digraph (N, E)
  - **Source node** $s \in N$ and **sink node** $t \in N$ $(s \neq t)$
  - Each edge (i, j) in E has a **length** c_{ij}
- The **length of a path** is the sum of the lengths of the edges in the path
- Problem: What is the shortest path from s to t?

Example 3. Consider the digraph below. The labels next to each edge represent that edge's length. What is the shortest path from node 1 to node 6?



- Natural applications of the shortest path problem:
  - Transportation (road networks, air networks)
  - Telecommunications (computer networks)
- Our focus: not-so-obvious applications of the shortest path problem

- In order to formulate a problem as a shortest path problem, we must specify:
  (i) a digraph (nodes and edges)
  (ii) a source and target node
  (iii) the length of each edge
  (iv) how any path from the source to the target translates into a solution to the problem

Example 4. You have just purchased a new car for $22,000. The cost of maintaining a car during a year depends on its age at the beginning of the year:

Age of car (years)	0	1	2	3	4
Annual maintenance cost ($)	2,000	3,000	4,000	8,000	12,000

To avoid the high maintenance costs associated with an older car, you may trade in your car and purchase a new car. The price you receive on a trade-in depends on the age of the car at the time of the trade-in:

Age of car (years)	1	2	3	4	5
Trade-in price ($)	15,000	12,000	9,000	5,000	2,000

For now, assume that at any time, it costs $22,000 to purchase a new car. Your goal is to minimize the net cost (purchasing costs + maintenance costs − money received in trade-ins) incurred over the next five years. Formulate your problem as a shortest path problem.

Example 5. The Simplexville College campus shuttle bus begins running at 7:00pm and continues until 2:00am. Several drivers will be used, but only one should be on duty at any time. If a shift starts at or before 9:00pm, a regular driver can be obtained for a 4-hour shift at a cost of $50. Otherwise, part-time drivers need to be used. Several part-time drivers can work 3-hours shifts at $40, and the rest are limited to 2-hour shifts at $30. The college's goal is to schedule drivers in a way that minimizes the total cost of staffing the shuttle bus. Formulate this problem as a shortest path problem.