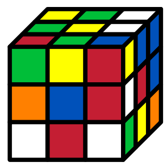
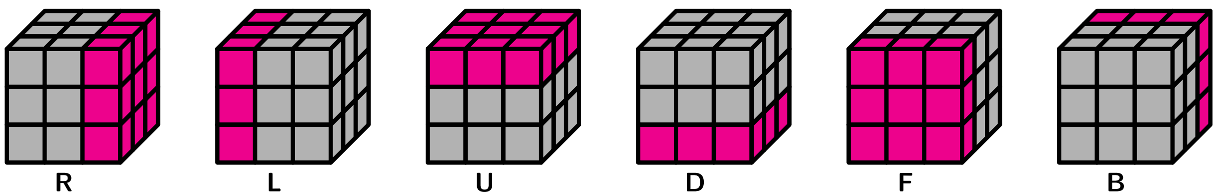
Lesson 7. Big DPs and the Curse of Dimensionality

1 Solving a Rubik's cube

- In a classic Rubik's cube, each of the 6 faces is covered by 9 stickers
- Each sticker can be one of 6 colors: white, red, blue, orange, green and yellow



- Each face of the cube can be turned independently
  - Notation:



  - The letter means turn the face clockwise 90°
    - For example, **R** means turn the right face clockwise 90°
  - The letter primed means turn the face counter-clockwise 90°
    - For example, **R'** means turn the right face counter-clockwise 90°
- The problem: given an initial configuration of the cube, find a *shortest* sequence of turns so that each face has only one color
  - You may assume that you are allowed at most T turns
  - It turns out that any configuration can be solved in 26 turns or less: `http://cube20.org/qtm/`
- How can we formulate this problem as a dynamic program?

- Stages:

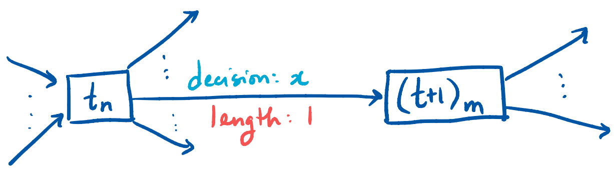
Stage t $\leftrightarrow$ t^{th} turn of the cube $(t = 1, \ldots, T)$

$\leftrightarrow$ end of decision-making process $(t = T+1)$

- States in stage t (nodes): Let $\underset{\text{initial}}{1}, \ldots, \underset{\text{solved}}{N}$ be a list of all the possible cube configurations

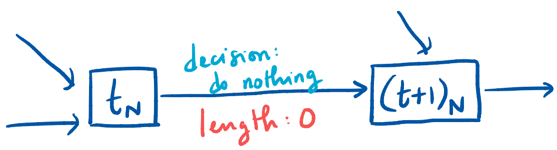
Node t_n $\leftrightarrow$ being in the n^{th} configuration with turns $t, t+1, \ldots, T$ remaining $(n = 1, \ldots, N)$

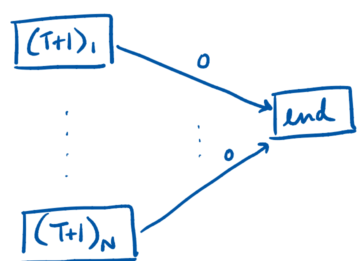
- Decisions, transitions, and rewards/costs at stage t (edges):



n^{th} configuration $\xrightarrow{\text{turn } x}$ m^{th} configuration

$x \in \{R, R', U, U', L, L', D, D', F, F', B, B'\}$





- Source node: 1_1 (initial config @ turn 1) Sink node: end

- Shortest/longest path? shortest

- Minimum number of turns required to solve the cube:

Length of a shortest path

- Actual sequence of turns that give the minimum number of turns to solve the cube:

Edges in the shortest path correspond to which turns to make.

2 Tetris

- You've all played Tetris before, right? Just in case...
- Tetris is a video game in which pieces fall down a 2D playing field, like this:

- Each piece is made up of four equally-sized bricks, and the playing field is 10 bricks wide and 20 bricks high
- As the pieces fall, the player can rotate them 90° in either direction, or move them left and right
- When a row is constructed without any holes, the player receives a point and the corresponding row is cleared
- The game is over once the height of bricks exceeds 20
- The problem: given a predetermined sequence of T pieces[1], determine how to place each piece in order to maximize the number of points accumulated over the course of the game
- How can we formulate this problem as a dynamic program?

[1] Normally, the sequence of falling pieces is random and infinitely long. We'll consider this easier version here.

- Stages:

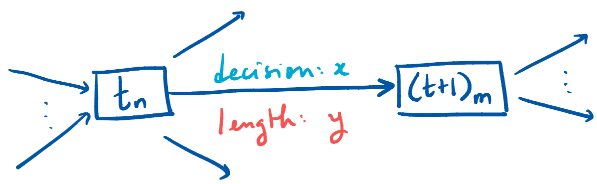
Stage t $\leftrightarrow$ playing the t^{th} piece $(t = 1, \dots, T)$
$\leftrightarrow$ end of the decision-making process $(t = T+1)$

- States in stage t (nodes): Let $1, \dots, N$ be a list of all the possible playing fields (1 = empty, N = full)

Node t_n $\leftrightarrow$ being in the n^{th} playing field with pieces $t, t+1, \dots, T$ remaining $(n = 1, \dots, N)$

- Decisions, transitions, and rewards/costs at stage t (edges):

n is not a losing playing field:

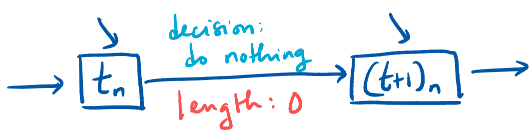


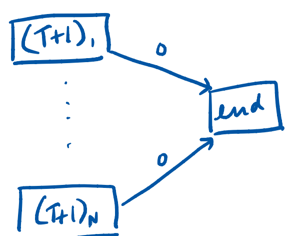
n^{th} playing field $\xrightarrow[\text{of piece } t]{\text{placement } x}$ m^{th} playing field

$x \in$ set of all possible placements of piece t in playing field n

$$y = \begin{cases} 1 & \text{if line is cleared with placement } x \text{ on playing field } n \text{ w/ piece } t \\ 0 & \text{o/w} \end{cases}$$

n is a losing playing field:





- Source node: 1_1 (empty field @ piece 1)   Sink node: end

- Shortest/longest path? longest

- Maximum number of points:

Length of a longest path

- Actual placement of pieces that give the maximum number of points:

Edges in a longest path correspond to which placements to make

3 Big DPs and the curse of dimensionality

- How big are these DPs we just formulated?
- Tetris:
  - Number of states per stage: $\boxed{N = 2^{200} \approx 1.61 \times 10^{60}}$
  - Number of stages T

  $\Rightarrow$ Number of nodes: $\boxed{N(T+1) + 1 \approx (1.61 \times 10^{60})(T+1) + 1}$
- Rubik's cube:
  - Number of states per stage: $\boxed{N \approx 4.33 \times 10^{19}}$
  - Number of stages T

  $\Rightarrow$ Number of nodes: $\boxed{N(T+1) + 1 \approx (4.33 \times 10^{19})(T+1) + 1}$
- The number of states is huge for both these DPs!

$\Rightarrow$ The DPs we formulated (as-is) are not solvable using today's computing power

- This is known as **the curse of dimensionality** in dynamic programming
- **Approximate dynamic programming** is an active area of research that tries to address the curse of dimensionality in various ways
  - For example, for Tetris: https://papers.nips.cc/paper/5190-approximate-dynamic-programming-finally-performs-well-in-the-game-of-tetris.pdf