B Solutions to Problems

Solution to Problem 1. First, let's define some notation to make things easier:

- Assign numbers to each phase:

$$1 \leftrightarrow \text{Research} \qquad 2 \leftrightarrow \text{Development}$$
$$3 \leftrightarrow \text{Manufacturing System Design} \qquad 4 \leftrightarrow \text{Initial Production and Distribution}$$

- Assign numbers to each speed:

$$0 \leftrightarrow \text{normal speed} \qquad 1 \leftrightarrow \text{priority speed}$$

- Define functions for the time required and the cost for each phase at the different speeds (i.e., the values in the given tables):

$$\begin{aligned} m(t,x) &= \text{months required for phase } t \text{ at speed } x \\ c(t,x) &= \text{cost for phase } t \text{ at speed } x \end{aligned} \qquad \text{for } t = 1,\dots,4 \text{ and } x \in \{0,1\}$$

Here is a dynamic program that models the problem:

- Stages:

$$t \leftrightarrow \begin{cases} \text{assign priority to phase } t & \text{if } t = 1,\dots,4 \\ \text{end of process} & \text{if } t = 5 \end{cases}$$

- States:

$$n \leftrightarrow n \text{ million dollars left with phases } t, t+1, \dots \text{ left to consider} \quad \text{for } n = 0,1,\dots,10$$

- Allowable decisions x_t at stage t and state n:
  - Stages $t = 1,2,3,4$: x_t represents the speed assigned to phase t. So, x_t must satisfy

$$x_t \in \{0,1\} \qquad c(t,x_t) \le n$$

  - Stage $t = 5$: no decisions

- Contribution of decision x_t at stage t and state n:

$$m(t,x_t) \qquad \text{for } t = 1,\dots,4 \text{ and } n = 0,1,\dots,10$$

- Value-to-go function:

$$f_t(n) = \text{minimum time to complete phases } t, t+1, \dots \text{ remaining with \$}n \text{ million budget}$$
$$\text{for } t = 1,\dots,5 \text{ and } n = 0,1,\dots,10$$

- Boundary conditions:

$$f_5(n) = 0 \qquad \text{for } n = 0,1,\dots,10$$

- Recursion:

$$f_t(n) = \min_{\substack{x_t \in \{0,1\} \\ c(t,x_t) \le n}} \Big\{ m(t,x_t) + f_{t+1}\big(n - c(t,x_t)\big) \Big\}$$

- Note that if $c(t,0) > n$ and $c(t,1) > n$, then it is impossible to complete phases $t, t+1, \dots$ with \$$n$ million, and so the minimum time $f_t(n)$ in this case is infinite.

- Desired value-to-go function value: $f_1(10)$

Now, let's solve this dynamic program by working backwards:

- Stage 5 computations – boundary conditions:

$$f_5(n) = 0 \quad \text{for } n = 0, 1, \dots, 10$$

- Stage 4 computations:

$$\begin{aligned}
f_4(10) &= \min\{2 + f_5(10-1), 1 + f_5(10-2)\} = \min\{2+0, 1+0\} = 1\\
f_4(9) &= \min\{2 + f_5(9-1), 1 + f_5(9-2)\} = \min\{2+0, 1+0\} = 1\\
f_4(8) &= \min\{2 + f_5(8-1), 1 + f_5(8-2)\} = \min\{2+0, 1+0\} = 1\\
f_4(7) &= \min\{2 + f_5(7-1), 1 + f_5(7-2)\} = \min\{2+0, 1+0\} = 1\\
f_4(6) &= \min\{2 + f_5(6-1), 1 + f_5(6-2)\} = \min\{2+0, 1+0\} = 1\\
f_4(5) &= \min\{2 + f_5(5-1), 1 + f_5(5-2)\} = \min\{2+0, 1+0\} = 1\\
f_4(4) &= \min\{2 + f_5(4-1), 1 + f_5(4-2)\} = \min\{2+0, 1+0\} = 1\\
f_4(3) &= \min\{2 + f_5(3-1), 1 + f_5(3-2)\} = \min\{2+0, 1+0\} = 1\\
f_4(2) &= \min\{2 + f_5(2-1), 1 + f_5(2-2)\} = \min\{2+0, 1+0\} = 1\\
f_4(1) &= \min\{2 + f_5(1-1)\} = 2\\
f_4(0) &= +\infty
\end{aligned}$$

- Stage 3 computations:

$$\begin{aligned}
f_3(10) &= \min\{5 + f_4(10-3), 3 + f_4(10-4)\} = \min\{5+1, 3+1\} = 4\\
f_3(9) &= \min\{5 + f_4(9-3), 3 + f_4(9-4)\} = \min\{5+1, 3+1\} = 4\\
f_3(8) &= \min\{5 + f_4(8-3), 3 + f_4(8-4)\} = \min\{5+1, 3+1\} = 4\\
f_3(7) &= \min\{5 + f_4(7-3), 3 + f_4(7-4)\} = \min\{5+1, 3+1\} = 4\\
f_3(6) &= \min\{5 + f_4(6-3), 3 + f_4(6-4)\} = \min\{5+1, 3+1\} = 4\\
f_3(5) &= \min\{5 + f_4(5-3), 3 + f_4(5-4)\} = \min\{5+1, 3+2\} = 5\\
f_3(4) &= \min\{5 + f_4(4-3), 3 + f_4(4-4)\} = \min\{5+2, 3+\infty\} = 7\\
f_3(3) &= \min\{5 + f_4(3-3)\} = 5 + \infty = +\infty\\
f_3(2) &= +\infty\\
f_3(1) &= +\infty\\
f_3(0) &= +\infty
\end{aligned}$$

- Stage 2 computations:

$$\begin{aligned}
f_2(10) &= \min\{3 + f_3(10-2), 2 + f_3(10-3)\} = \min\{3+4, 2+4\} = 6\\
f_2(9) &= \min\{3 + f_3(9-2), 2 + f_3(9-3)\} = \min\{3+4, 2+4\} = 6\\
f_2(8) &= \min\{3 + f_3(8-2), 2 + f_3(8-3)\} = \min\{3+4, 2+5\} = 7\\
f_2(7) &= \min\{3 + f_3(7-2), 2 + f_3(7-3)\} = \min\{3+5, 2+7\} = 8\\
f_2(6) &= \min\{3 + f_3(6-2), 2 + f_3(6-3)\} = \min\{3+7, 2+\infty\} = 10
\end{aligned}$$

$$
\begin{aligned}
f_2(5) &= \min\{3 + f_3(5-2), 2 + f_3(5-3)\} = \min\{3+\infty, 2+\infty\} = +\infty \\
f_2(4) &= \min\{3 + f_3(4-2), 2 + f_3(4-3)\} = \min\{3+\infty, 2+\infty\} = +\infty \\
f_2(3) &= \min\{3 + f_3(3-2), 2 + f_3(3-3)\} = \min\{3+\infty, 2+\infty\} = +\infty \\
f_2(2) &= \min\{3 + f_3(2-2)\} = 3 + \infty = +\infty \\
f_2(1) &= +\infty \\
f_2(0) &= +\infty
\end{aligned}
$$

- Stage 1 computations – desired value-to-go:

$$f_1(10) = \min\{4 + f_2(10-2), 2 + f_2(10-3)\} = \min\{4+7, 2+8\} = 10$$

- Tracing through the recursion, we see that we should run phases 1 and 3 at priority speed, and phases 2 and 4 at normal speed.

Solution to Problem 2. First, let's define some notation to make things easier. Let

$$
\begin{aligned}
c_t &= \text{cost of producing a laptop in month } t \quad && \text{for } t = 1,2,3 \\
d_t &= \text{demand for laptops in month } t && \text{for } t = 1,2,3
\end{aligned}
$$

Here is a dynamic program that models the problem:

- Stages:

$$t \leftrightarrow \begin{cases} \text{beginning of month } t & \text{if } t = 1,\dots,3 \\ \text{end of process} & \text{if } t = 4 \end{cases}$$

- States:

$$n \leftrightarrow n \text{ laptops in inventory} \quad \text{for } n = 0, 100, 200, 300, 400$$

- Allowable decisions x_t at stage t and state n:
  - Stages $t = 1,2,3$: x_t represents the number of laptops to produce. So, x_t must satisfy

$$x_t \in \{0, 100, 200, 300\} \qquad 0 \le n + x_t - d_t \le 400$$

  - Stage $t = 4$: no decisions

- Contribution of decision x_t at stage t and state n:

$$2500 I(x_t) + c_t x_t + 15(n + x_t - d_t) \qquad \text{for } t = 1,2,3 \text{ and } n = 0, 100, 200, 300, 400$$

  where

$$I(x_t) = \begin{cases} 1 & \text{if } x_t > 0 \\ 0 & \text{otherwise} \end{cases}$$

- Value-to-go function:

$$f_t(n) = \text{minimum cost of meeting demand in months } t, t+1, \dots \text{ with an initial inventory of } n \text{ laptops}$$
$$\text{for } t = 1,\dots,4 \text{ and } n = 0, 100, 200, 300, 400$$

- Boundary conditions:

$$f_4(n) = 0 \qquad \text{for } n = 0, 100, 200, 300, 400$$

- Recursion:

$$f_t(n) = \min_{\substack{x_t \in \{0,100,200,300\} \\ 0 \le n + x_t - d_t \le 400}} \left\{2500I(x_t) + c_t x_t + 15(n + x_t - d_t) + f_{t+1}(n + x_t - d_t)\right\}$$
$$\text{for } t = 1, 2, 3 \text{ and } n = 0, 100, 200, 300, 400$$

- Desired value-to-go function value: $f_1(0)$

Now, let's solve this dynamic program by working backwards:

- Stage 4 computations – boundary conditions:

$$f_4(n) = 0 \quad \text{for } n = 0, 100, 200, 300, 400$$

- Stage 3 computations:

$$\begin{aligned}
f_3(400) &= \min\{15(200) + f_4(200), 2500 + 120(100) + 15(300) + f_4(300), 2500 + 120(200) + 15(400) + f_4(400)\} \\
&= \min\{3000, 19000, 32500\} = 3000 \\
f_3(300) &= \min\{15(100) + f_4(100), 2500 + 120(100) + 15(200) + f_4(200), 2500 + 120(200) + 15(300) + f_4(300), \\
&\qquad 2500 + 120(300) + 15(400) + f_4(400)\} \\
&= \min\{1500, 17500, 31000, 44500\} = 1500 \\
f_3(200) &= \min\{15(0) + f_4(0), 2500 + 120(100) + 15(100) + f_4(100), 2500 + 120(200) + 15(200) + f_4(200), \\
&\qquad 2500 + 120(300) + 15(300) + f_4(300)\} \\
&= \min\{0, 16000, 29500, 43000\} = 0 \\
f_3(100) &= \min\{2500 + 120(100) + 15(0) + f_4(0), 2500 + 120(200) + 15(100) + f_4(100), \\
&\qquad 2500 + 120(300) + 15(200) + f_4(200)\} \\
&= \min\{14500, 28000, 41500\} = 14500 \\
f_3(0) &= \min\{2500 + 120(200) + 15(0) + f_4(0), 2500 + 120(300) + 15(100) + f_4(100)\} \\
&= \min\{26500, 40000\} = 26500
\end{aligned}$$

- Stage 2 computations:

$$\begin{aligned}
f_2(400) &= \min\{15(100) + f_3(100), 2500 + 100(100) + 15(200) + f_3(200), \\
&\qquad 2500 + 100(200) + 15(300) + f_3(300), 2500 + 100(300) + 15(400) + f_3(400)\} \\
&= \min\{16000, 15500, 28500, 41500\} = 15500 \\
f_2(300) &= \min\{15(0) + f_3(0), 2500 + 100(100) + 15(100) + f_3(100), \\
&\qquad 2500 + 100(200) + 15(200) + f_3(200), 2500 + 100(300) + 15(300) + f_3(300)\} \\
&= \min\{26500, 28500, 25500, 38500\} = 25500 \\
f_2(200) &= \min\{2500 + 100(100) + 15(0) + f_3(0), \\
&\qquad 2500 + 100(200) + 15(100) + f_3(100), 2500 + 100(300) + 15(200) + f_3(200)\} \\
&= \min\{39000, 38500, 35500\} = 35500 \\
f_2(100) &= \min\{2500 + 100(200) + 15(0) + f_3(0), 2500 + 100(300) + 15(100) + f_3(100)\} \\
&= \min\{49000, 48500\} = 48500 \\
f_2(0) &= \min\{2500 + 100(300) + 15(0) + f_3(0)\} = 59000
\end{aligned}$$

- Stage 1 computations – desired value-to-go:

$$\begin{aligned}
f_1(0) &= \min\{2500 + 100(200) + 15(0) + f_2(0), 2500 + 100(300) + 15(100) + f_2(100)\} \\
&= \min\{81500, 82500\} = 81500
\end{aligned}$$

- Tracing through the recursion, we see that we should produce 200 in month 1, 300 in month 2, and 200 in month 3.