SA402 – Dynamic and Stochastic Models Fall 2013
Asst. Prof. Nelson Uhan

Exam 2 – Information and Review Problems

1 Information

- When: **Thursday October 31** in class
- What: Lessons 8 – 17
- One 8.5 in × 11 in sheet of handwritten notes allowed
- Calculators allowed
- Review on Tuesday October 29 in class
- EI on Wednesday October 30, 1900 – 2030, CH348

2 Review Problems

Problem 1. A fisherman catches fish according to a Poisson process with rate $\lambda = 0.6$ fish per hour. The fisherman will keep fishing for two hours. If he has caught at least one fish, he quits. Otherwise, he continues until he catches at least one fish.

a. What is the probability that he stays for more than two hours?
b. What is the probability that he catches no fish in the first two hours, given that he did not catch any fish in the first hour?
c. What is the probability that he catches at least two fish?
d. What is the expected number of fish that he catches in the first two hours?
e. What is his expected total fishing time, given that he has been fishing for four hours?
f. (harder) What is the expected number of fish that he catches?

Problem 2. Transmitters A and B independently send messages to a single receiver according to a Poisson process, with rates λ_A and λ_B, respectively. All messages are so brief that we may assume that they occupy single points in time. The number of words in a message, regardless of the source that is transmitting it, is a random variable W such that

$$\Pr\{W = 1\} = 2/6 \qquad \Pr\{W = 2\} = 3/6 \qquad \Pr\{W = 3\} = 1/6 \tag{1}$$

a. What assumptions are implicitly being made by stating that the messages from either transmitter arrive according to a Poisson process?
b. What is the probability that during an interval of duration t, a total of exactly nine messages will be received?
c. What is the probability that exactly eight of the next twelve messages received will be from transmitter A?
d. What is the expected total number of 2 word messages?
e. (harder) What is the expected total number of words received during an interval of duration t?

Problem 3. The Poisson Seafood Market opens at 8 a.m. and closes at 6 p.m. From 8 a.m. to 10 a.m., customers arrive according to a Poisson process (of course) at a rate of 4 per hour. From 10 a.m. to 4 p.m., the arrival rate increases to 6 per hour, and finally, from 4 p.m. to 6 p.m., the arrival rate increases to 10 per hour.

a. Let A be a random variable that represents the number of customers that entered the market on a given day. What distribution does A follow? Be specific.
b. If there has been a total of 30 customers by 2 p.m, what is the probability that the total number of customers by closing is more than 60?
c. What is the expected number of customers from 8 a.m. to 12 p.m.?

Problem 4. Professor May B. Wright gives quizzes that are hard, medium, or easy. If she gives a hard quiz, her next quiz will be either medium or easy, with equal probability. However, if she gives a medium or easy quiz, there is a 0.5 probability that her next quiz will be of the same difficulty, and a 0.25 probability for each of the other two levels of difficulty.

a. Model Professor Wright's quiz difficulty as a Markov chain by specifying the state space and one-step transition probability matrix.
b. Suppose Professor Wright's SA 402 Quiz 1 was hard. What is the probability that Quiz 4 will be easy?
c. Again, suppose Quiz 1 was hard. What is the probability that the next 3 quizzes will be hard or medium, and Quiz 5 will be easy?
d. Classify the states in your Markov chain as transient or recurrent. Explain how you obtained your classification.
e. In the long run, what fraction of Professor Wright's quizzes are easy? Medium? Hard?

Problem 5. Consider a Markov chain with state space $\mathcal{M} = \{1, 2, 3, 4, 5\}$ and the following one-step transition probability matrix:

$$\mathbf{P} = \begin{pmatrix} 0 & 0.5 & 0 & 0.2 & 0.3 \\ 0.5 & 0 & 0.1 & 0.1 & 0.3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 0.6 & 0.4 \end{pmatrix}$$

a. Draw the transition-probability diagram.
b. Explain why $\{3\}$ and $\{4, 5\}$ are irreducible sets.
c. What is the probability that the process ends up in state 3 in the long run, given that it starts at state 1?
d. (harder) What is the probability that the process ends up in states 4 <u>or</u> 5, given that it starts at state 1?

Problem 6. Professor I. M. Wright would like to model the weather conditions in Simplexville. The weather can roughly be categorized as "sunny" or "cloudy". According to the meteorological experts, the weather on any given day depends on the weather conditions during the previous two days.

Professor Wright proposes to model the weather conditions in the future as a Markov chain with two states: 1 for "sunny", and 2 for "cloudy." Is this a good idea? Why or why not? If not, propose a modification that will allow him to model this setting as a Markov chain: (1) specify the state space in detail, (2) specify the meaning of the nth state in the case's context, and (3) describe the meaning of the one-step transition probabilities and initial state probabilities in the case's context.