Problem 1 (Nelson 6.11, modified)**.** A food manufacturer plans to introduce a new potato chip, Box O' Spuds, into a local market that already has three strong competitors. Because the manufacturer makes a superior product, the marketing analysts believe that brand loyalty to Box O' Spuds will be high after people try it. However, it will be difficult to persuade people to switch from the three established brands. The marketing analysts would like to forecast the long-term market share for Box O' Spuds to determine whether it is worth entering the market.

Suppose the marketing analysts formulate a Markov chain model of customer brand switching in which the state space $\mathcal{M} = \{1, 2, 3, 4\}$ corresponds to which of the three established brands or Box O' Spuds, respectively, that a customer currently purchases. The time index is the number of bags of chips purchased. Based on market research and experience with other products, the one-step transition matrix the marketing analysts anticipate is

$$\mathbf{P} = \begin{pmatrix} 0.70 & 0.14 & 0.14 & 0.02 \\ 0.14 & 0.70 & 0.14 & 0.02 \\ 0.14 & 0.14 & 0.70 & 0.02 \\ 0.05 & 0.05 & 0.05 & 0.85 \end{pmatrix}$$

a. Describe in words what the entries of $\mathbf{P}$ represent. For example, what does $p_{32} = 0.14$ mean?

b. Suppose that initially, a typical customer is equally likely to prefer one of the three existing brands. What is the probability that a typical customer prefers Box O' Spuds after he or she has bought 100 bags of chips?

a. The entries of $\mathbb{P}$ represent how a customer's preference of potato chip brand changes for each bag purchased. In particular,

$$p_{ij} = \Pr\{\text{next bag is brand } j \mid \text{previous bag was brand } i\}$$

b. Let the initial state vector be $\vec{p} = (\frac{1}{3} \ \ \frac{1}{3} \ \ \frac{1}{3} \ \ 0)^T$.
This reflects the fact that a customer is equally likely to buy one of the three existing brands.

We want $p_4^{(100)}$. We compute $\vec{p}^{(100)^T} = \vec{p}^T \mathbb{P}^{100} = (\frac{1}{3} \ \ \frac{1}{3} \ \ \frac{1}{3} \ \ 0)\begin{pmatrix} 0.70 & 0.14 & 0.14 & 0.02 \\ 0.14 & 0.70 & 0.14 & 0.02 \\ 0.14 & 0.14 & 0.70 & 0.02 \\ 0.05 & 0.05 & 0.05 & 0.85 \end{pmatrix}^{100}$

$$\approx (0.294 \ \ 0.294 \ \ 0.294 \ \ 0.118)$$

So, $p_4^{(100)} \approx 0.118$