Lesson 1. Sample Paths

1 Course overview

- In this course, we will learn how to model and analyze systems that evolve <u>dynamically</u> over time and whose behavior is <u>stochastic</u>, or uncertain

- The models we will use to analyze such systems are called **stochastic processes**

- Numerous applications:

    - manufacturing - capacity planning, inventory control
    - business and economics - consumer behavior, portfolio management
    - biology - population genetics, spread of epidemics
    - military - combat strategy, ordnance effectiveness

- Techniques used: probability, statistics, linear algebra, basic notions of simulation

- Today: one example illustrating some basic ideas

2 The Case of the Copy Enlargement

The Darker Image, a national chain of small photocopying shops, currently configures each store with one photocopying machine and one clerk. Arriving customers stand in a single line to wait for the clerk. The clerk completes the customers' photocopying jobs one at a time, first-come-first-served, including collecting payment for the job.

Business is good enough that company headquarters plans to enlarge some stores by adding a second photocopying machine. The second copier could be operated by a new clerk, but since some customers with small copying jobs have complained about having to wait a long time, the company is considering installing the second copier for self-service copying only. The company wants to know which option will deliver better service.

- Some things need to be clarified, including:

    - How many customers will use a self-service copier?
    - Can a self-service copier provide all the services that a customer wants?
    - How do we measure "better service"?

- Important observation: neither option will be the best option all the time, since the system (and its performance) is subject to uncertainty, such as:

- We have some data collected one morning at a Darker Image location:

Time	What happened	Comment	Time	What happened	Comment
0		shop opens	76	customer 12 arrived	
12	customer 1 arrived		79	customer 13 arrived	
14	customer 2 arrived		84	customer 10 finished	two-sided, staple
17	customer 3 arrived		86	customer 11 finished	
19	customer 1 finished	collate, staple	88	customer 14 arrived	
21	customer 2 finished		96	customer 12 finished	
22	customer 3 finished		96	customer 15 arrived	
38	customer 4 arrived		98	customer 13 finished	
39	customer 5 arrived		99	customer 14 finished	
41	customer 4 finished		102	customer 15 finished	collate, staple
43	customer 6 arrived		105	customer 16 arrived	
45	customer 7 arrived		107	customer 16 finished	
52	customer 8 arrived		108	customer 17 arrived	
57	customer 5 finished	special paper	110	customer 17 finished	
58	customer 9 arrived		113	customer 18 arrived	
60	customer 6 finished		118	customer 19 arrived	
61	customer 7 finished		120	customer 20 arrived	
68	customer 8 finished		121	customer 18 finished	special paper
71	customer 9 finished	collate, covers	127	customer 19 finished	two-sided, staple
73	customer 10 arrived		129	customer 20 finished	collate, staple
74	customer 11 arrived				

  - Time = minutes elapsed after 9:00
  - Customer numbers assigned in order of their arrival

3   Analyzing the service times

- Company HQ thinks any customer whose job doesn't need special handling (i.e. collating, stapling, two-sided, etc.) could have potentially used a self-service copier

- A reasonable first question: are the **service times** – the times required to complete the copying jobs – different for "regular" customers (jobs without special handling) compared to "special" customers (jobs with special handling)?

- Service time for customer 1:
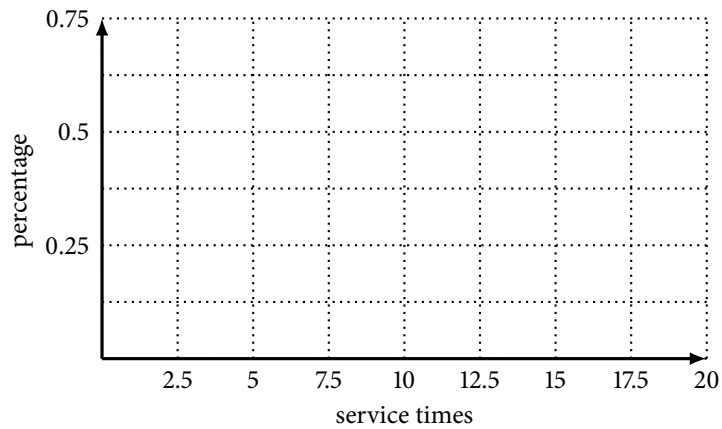
- Service time for customer 2:

- More generally: for $i = 1, \ldots, 20$

  - Let a_i = arrival time for customer i
 - Let d_i = departure (finish) time for customer i
  - Service time for customer i:

- Service times for customers (* indicates special handling):

Customer number (i)	Service time (s_i)	Customer number (i)	Service time (s_i)
1*		11	
2		12	
3		13	
4		14	
5*		15*	
6		16	
7		17	
8		18*	
9*		19*	
10*		20*	

- Histogram of service times for "regular" customers (jobs without special handling):

- Histogram for service times for "special" customers (jobs with special handling):

3
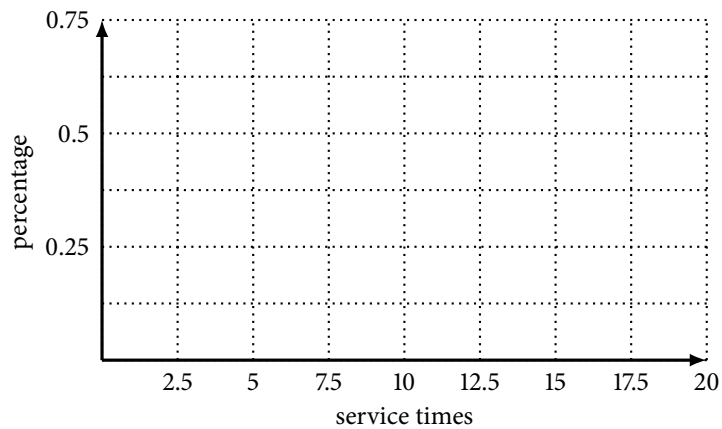
- So, given our analysis of the service times:

  - Do the distributions look different?
  - Should the company go with the self-service option?

- A better performance measure: **delay**, or the time a customer spends waiting in the **queue** (the line)

- How will adding a self-service copier or a second full-service copier affect the delay for regular and special customers?

- To measure delay, we need to consider the dynamics of the system

  - The delay of one customer depends on the actions of other customers

4 Sample path decomposition and analysis

- A **sample path** is a record of the time-dependent behavior of a system

  - For example, the time-study data on page 2

- A sample path can be decomposed into **inputs** and **logic**, for example:

  - Inputs = arrival times and service times of customers
  - Logic = how the shop operates – number of copiers, first-come-first-served, etc.

- **Simulation** generates new sample paths resulting from changes in the input or logic without building the new system

- **Sample-path analysis** extracts system performance measures from sample paths

- The Darker Image is interested in changing the logic of the system, while the inputs are out of its control

- To compare customer delay when adding a self-service copier vs. a second full-service copier, we will simulate both systems and analyze the resulting sample paths

5 Interarrival times

- Using interarrival times instead of arrival times is a common convention

- The **interarrival time** of customer i is the time gap between the arrival of customers $i-1$ and i:

- Interarrival times for customers (* indicates special handling):

Customer number (i)	Interarrival time (g_i)	Customer number (i)	Interarrival time (g_i)
1*	12	11	1
2	2	12	2
3	3	13	3
4	21	14	9
5*	1	15*	8
6	4	16	9
7	2	17	3
8	7	18*	5
9*	6	19*	5
10*	15	20*	2

6   Simulating the self-service system

- Let's start by simulating the proposed self-service system

- Assumptions on system logic:

  - 2 queues: full service queue, self-service queue
  - Regular customers always join the self-service queue, specials always join the full-service queue
  - Customers do not change queues after they join one
  - Service times are the same as in the original one-copier system

- To simulate the system, we need to keep track of the following **system events**:

  - a customer arrives
  - a job finishes at the full-service copier
  - a job finishes at the self-service copier

- Let's simulate the system until the arrival of customer 10

- After simulating all 20 customers (for homework), you should find that

  - the simulation ends after 129 minutes
  - regular customers 7 and 13 experience delays of 1 and 7 minutes, respectively
  - special customers 19 and 20 experience delays of 3 and 7 minutes, respectively
  - all other customers have delays of 0 minutes

7   Simulating the full-service system

- We can do something similar for the proposed full-service system

- Assumptions on system logic:

  - 1 queue for both copiers/clerks
  - Service delivered first-come-first-served by next available copier/clerk

- After simulating all 20 customers (for homework), you should find that

  - the simulation ends after 124 minutes
  - regular customers 7 and 13 experience delays of 1 and 5 minutes, respectively
  - special customer 20 experiences a delay of 1 minute
  - all other customers have delays of 0 minutes

8   Comparisons

- Based on these simulations, the full-service system is superior for both types of customers

- How much confidence should we have in these conclusions?

- Concern:

    ○ We measured delay based on only <u>one</u> possible sample path for each proposed system
    ○ Better: a <u>distribution</u> of delay based on <u>many</u> possible sample paths

- One way to do this would be to obtain more data and perform more simulations – focus of SA421

- If we make certain assumptions (often reasonable), we can mathematically analyze sample paths without actually generating them – focus of this course

Current time:

Full Service		Self-Service
Next system event		**Time**
customer arrival		
full-service finish		
self-service finish		

Current time:

Full Service		Self-Service
Next system event		**Time**
customer arrival		
full-service finish		
self-service finish		

Current time:

Full Service		Self-Service
Next system event		**Time**
customer arrival		
full-service finish		
self-service finish		

Current time:

Full Service		Self-Service
Next system event		**Time**
customer arrival		
full-service finish		
self-service finish		

Current time:

Full Service		Self-Service
Next system event		**Time**
customer arrival		
full-service finish		
self-service finish		

Current time:

Full Service		Self-Service
Next system event		**Time**
customer arrival		
full-service finish		
self-service finish		

Current time:

Full Service		Self-Service
Next system event		**Time**
customer arrival		
full-service finish		
self-service finish		

Current time:

Full Service		Self-Service
Next system event		**Time**
customer arrival		
full-service finish		
self-service finish		

Current time:

Full Service		Self-Service
Next system event		**Time**
customer arrival		
full-service finish		
self-service finish		

Current time:

Full Service		Self-Service
Next system event		**Time**
customer arrival		
full-service finish		
self-service finish		

Current time:

Full Service		Self-Service
Next system event		**Time**
customer arrival		
full-service finish		
self-service finish		

Current time:

Full Service		Self-Service
Next system event		**Time**
customer arrival		
full-service finish		
self-service finish		

Current time:

Full Service		Self-Service
Next system event		**Time**
customer arrival		
full-service finish		
self-service finish		

Current time:

Full Service		Self-Service
Next system event		**Time**
customer arrival		
full-service finish		
self-service finish		

Current time:

Full Service		Self-Service
Next system event		**Time**
customer arrival		
full-service finish		
self-service finish		

Current time:

Full Service		Self-Service
Next system event		**Time**
customer arrival		
full-service finish		
self-service finish		

Current time:

Full Service		Self-Service
Next system event		**Time**
customer arrival		
full-service finish		
self-service finish		

Current time:

Full Service		Self-Service
Next system event		**Time**
customer arrival		
full-service finish		
self-service finish		

Current time:

Full Service		Self-Service
Next system event		**Time**
customer arrival		
full-service finish		
self-service finish		

Current time:

Full Service		Self-Service
Next system event		**Time**
customer arrival		
full-service finish		
self-service finish		