Lesson 3. Conditional Probability Review

0 Warm up

Example 1. The Markov Company sells three types of replacement wheels and two types of bearings for in-line skates. Wheels and bearings must be ordered as a set, but customers can decide which combination of wheel type and bearing type they want.

Let V and W be random variables that represent the type of bearing and wheel, respectively, in replacement sets ordered in the future. Based on historical data, the company has determined the probability that each wheels-bearings pair will be ordered:

		W		
		1	2	3
V	1	2/10	1/10	1/10
	2	1/20	8/20	3/20

The table above defines the **joint probability mass function (pmf)** p_{VW} of V and W, where $p_{VW}(a,b) = \Pr\{V = a, W = b\}$. For example, $p_{VW}(1,2) = \Pr\{V = 1, W = 2\} = 1/10$.

a. What is $\Pr\{V = 1\}$?
b. What is $\Pr\{W = 2\}$?

1 Last time: independence

- Let X and Y be random variables with joint cdf F_{XY} and marginal cdfs F_X and F_Y
- X and Y are independent if

$$F_{XY}(a,b) = \Pr\{X \le a, Y \le b\} = \Pr\{X \le a\}\Pr\{Y \le b\} = F_X(a)F_Y(b)$$

- More generally:

  A joint probability statement about any finite number of independent random variables decomposes into the product of the marginal probability statements.

- Are V and W in Example 1 independent?

2 Conditional probability

- Let V and W be two random variables
- **Conditional probability** addresses the question:

  How should we revise our probability statements about W given that we have some knowledge of the value of V?

- Let's consider events of the form $\{V \in \mathcal{A}\}$ and $\{W \in \mathcal{B}\}$, for example:
  - $\mathcal{A} = (5, 27] \Rightarrow \{V \in \mathcal{A}\} = \{5 < V \leq 27\}$
  - $\mathcal{B} = \{44, 73\} \Rightarrow \{W \in \mathcal{B}\} = \{V = 44 \text{ or } V = 73\}$
  - $\{V = a\}$ can be written as $\{V \in \mathcal{A}\}$ with
  - $\{W \leq b\}$ can be written as $\{W \in \mathcal{B}\}$ with
- The conditional probability that W takes a value in $\mathcal{B}$ given that V takes a value in $\mathcal{A}$ is:
  - The revised probability is the probability of the joint event $\{W \in \mathcal{B}, V \in \mathcal{A}\}$ normalized by the probability of the conditional event $\{V \in \mathcal{A}\}$

Example 2. In Example 1, what is the probability that a customer will order type 2 wheels, given that he or she orders type 1 bearings?

- If V and W are independent, then:
- Let $\mathcal{A} \subseteq \mathcal{B}$. Then, if V and W are perfectly dependent (i.e., $V = W$), then:

3 Conditional distributions and expectations

- Let V and W be discrete random variables
  - In particular, suppose W takes on values $b_1, b_2, \ldots$
- The **conditional probability mass function (pmf)** of W given $V \in \mathcal{A}$ is:

- The **conditional cumulative distribution function (cdf)** of W given $V \in \mathcal{A}$ is:

- The **conditional expected value** of $g(W)$ given $V \in \mathcal{A}$ is:

Example 3. In Example 1, find the conditional pmf of W given that $V = 1$.

Example 4. In Example 1, suppose that the profit from selling type 1, type 2, and type 3 wheels is \$4, \$6, and \$10, respectively. Find the expected profit from wheels, given that the customer ordered type 1 bearings.

4 Law of total probability

- We can write joint probabilities as products of conditional probabilities and marginal probabilities:

- We can also decompose marginal probabilities: **the law of total probability**
  - Suppose V is a discrete r.v. taking values $a_1, a_2, \dots$. Then:

Example 5. In Example 1, the conditional pmf of W given that $V = 2$ is:

b	1	2	3
$p_{W\|V=2}(b)$	1/12	8/12	3/12

Use this with your answer to Example 3 to find $\Pr\{W = 2\}$.

Example 6 (Nelson 3.6)**.** Let Y be an exponential random variable with parameter $\lambda = 2$ that models the time to deliver a pizza in hours.

a. The pizza-delivery company promises delivery within 40 minutes, or the pizza is free. What is the probability that we will get a free pizza, given that we have already waited 30 minutes?
b. What is the probability that delivery takes longer than 1 hour, given it takes longer than 40 minutes?
c. What is the probability that delivery takes less than 10 minutes, given it takes less than 40 minutes?

Example 7. James plays a game in which his score is an integer number from 1 to 10, and that each of these 10 numbers is equally likely to be his score. The first time he plays the game, his score is X. He then continues to play the game until he obtains another score Y such that $Y \geq X$. Assume all plays of the game are independent. What is the probability that $Y = 10$?