SA402 – Dynamic and Stochastic Models — Fall 2013
Asst. Prof. Nelson Uhan

Lesson 4. Conditional Probability Review, cont.

Problem 1. James plays a game in which his score is an integer number from 1 to 5, and that each of these 5 numbers is equally likely to be his score. The first time he plays the game, his score is X. He then continues to play the game until he obtains another score Y such that $Y \geq X$. Assume all plays of the game are independent.

a. Find the probability that $X = x$, for any $x = 1, \dots, 5$.
b. Find the probability that $Y = 5$ given $X = x$.
c. Using your answers from parts a and b and the law of total probability, find the probability that $Y = 5$.

Problem 2. Melanie rolls a six-sided die, and records the number N on the top face. Then she tosses a fair coin N times, and records the total number Z of heads that appear.

a. Find the probability that $N = n$, for any $n = 1, \dots, 6$.
b. Find the probability that $Z = 2$ given $N = n$.
c. Using your answers from parts a and b and the law of total probability, find the probability that $Z = 2$.

Problem 3. Let X and Y be discrete random variables, with X taking on values $x_1, x_2, \dots$, and Y taking on values $y_1, y_2, \dots$. Show that

$$E[g(X)] = \sum_{\text{all } j} E[g(X) | Y = y_j] p_Y(y_j). \tag{1}$$

Hint: start with the expression on the right hand side, and use the law of total probability.

Equation (1) is sometimes referred to as the **law of total expectation**.

Problem 4. The Simplex Company uses three machines to produce a large batch of similar manufactured items. 20% of the items were produced by machine 1, 30% by machine 2, and 50% by machine 3. In addition, 1% of the items produced by machine 1 are defective, 2% by machine 2 are defective, and 3% by machine 3 are defective. Suppose you select 1 item at random from the entire batch.

a. Define the random variable M as the machine used ($M \in \{1, 2, 3\}$) to produce this item. Write the pmf p_M of M.
b. Define another random variable D that is equal to 1 if this item is defective, and 0 otherwise. Find the probability that $D = 1$ given $M = m$, for $m = 1, 2, 3$.
c. Using your answers from parts a and b and the law of total probability, find the probability that $D = 1$; that is, the probability that the randomly selected item is defective.