P1. a. $\Pr\{X=x\} = \frac{1}{5}$ for $x=1,\dots,5$, since each number is equally likely.

b. $\Pr\{Y=5 \mid X=x\} = \frac{1}{6-x}$, since $Y = x, x+1, \dots, 5$ are equally likely.

c. $$\Pr\{Y=5\} = \sum_{x=1}^{5} \Pr\{Y=5 \mid X=x\} \Pr\{X=x\} \quad \text{(law of total probability)}$$
$$= \sum_{x=1}^{5} \frac{1}{6-x} \cdot \frac{1}{5} = \frac{1}{5}\left(\frac{1}{5}+\frac{1}{4}+\frac{1}{3}+\frac{1}{2}+1\right) \approx 0.457.$$

P2. a. $\Pr\{N=n\} = \frac{1}{6}$ for $n=1,\dots,6$

b. If $n=1$, $\Pr\{Z=2 \mid N=n\} = 0$.

If $n \geq 2$, $\Pr\{Z=2 \mid N=n\} = \binom{n}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{n-2} = \binom{n}{2}\left(\frac{1}{2}\right)^n$

c. $$\Pr\{Z=2\} = \sum_{n=1}^{6} \Pr\{Z=2 \mid N=n\} \Pr\{N=n\} \quad \text{(law of total probability)}$$
$$= 0 + \sum_{n=2}^{6} \binom{n}{2}\left(\frac{1}{2}\right)^n\left(\frac{1}{6}\right) = \frac{1}{6}\left(1\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + 6\left(\frac{1}{2}\right)^4 + 10\left(\frac{1}{2}\right)^5 + 15\left(\frac{1}{2}\right)^6\right)$$
$$\approx 0.258$$

P3. $$\sum_{\text{all } j} E[g(X) \mid Y=y_j]\, p_Y(y_j) = \sum_{\text{all } j} \sum_{\text{all } i} g(x_i)\, p_{X|Y=y_j}(x_i)\, p_Y(y_j)$$
$$= \sum_{\text{all } j} \sum_{\text{all } i} g(x_i) \Pr\{X=x_i \mid Y=y_j\} \Pr\{Y=y_j\} = \sum_{\text{all } i} g(x_i) \sum_{\text{all } j} \Pr\{X=x_i \mid Y=y_j\} \Pr\{Y=y_j\}$$
$$\underset{\substack{\text{law of total}\\ \text{probability}}}{=} \sum_{\text{all } i} g(x_i) \Pr\{X=x_i\} = \sum_{\text{all } i} g(x_i)\, p_X(x_i) = E[g(X)]$$

P4. a. $$\underset{\Pr\{M=m\}}{\overset{}{p_M(m)}} = \begin{cases} 0.2 & \text{if } m=1 \\ 0.3 & \text{if } m=2 \\ 0.5 & \text{if } m=3 \end{cases}$$

b. $\Pr\{D=1 \mid M=1\} = 0.01$
$\Pr\{D=1 \mid M=2\} = 0.02$
$\Pr\{D=1 \mid M=3\} = 0.03$

c. $$\Pr\{D=1\} = \sum_{m=1}^{3} \Pr\{D=1 \mid M=m\} \Pr\{M=m\}$$
$$= 0.01(0.2) + 0.02(0.3) + 0.03(0.5)$$
$$= 0.023.$$