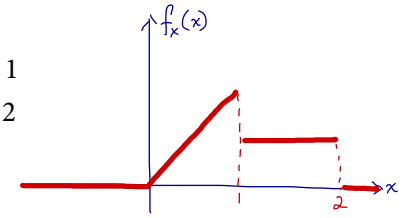
Example 3. Let X be a random variable with pdf

$$f_X(a) = \begin{cases} 0 & \text{if } x < 0 \\ a & \text{if } 0 \le a \le 1 \\ \frac{1}{2} & \text{if } 1 < a \le 2 \\ 0 & \text{if } a > 2 \end{cases}$$



a. Find the cdf of X.
b. Find a random variate generator for X.

a. $F_x(a) = \int_{-\infty}^{a} f_x(b)\,db$

If $a<0$: $F_x(a) = \int_{-\infty}^{a} 0\,db = 0$

If $0 \le a \le 1$: $F_x(a) = \int_{-\infty}^{0} 0\,db + \int_{0}^{a} b\,db = 0 + \left[\frac{b^2}{2}\right]_0^a = \frac{a^2}{2}$

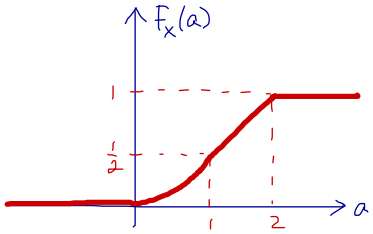
If $1 < a \le 2$: $F_x(a) = \int_{-\infty}^{0} 0\,db + \int_{0}^{1} b\,db + \int_{1}^{a} \frac{1}{2}\,db = 0 + \left[\frac{b^2}{2}\right]_0^1 + \left[\frac{b}{2}\right]_1^a$

$= \frac{1}{2} + \frac{a}{2} - \frac{1}{2} = \frac{a}{2}$

If $a>2$: $F_x(a) = \int_{-\infty}^{0} 0\,db + \int_{0}^{1} b\,db + \int_{1}^{2} \frac{1}{2}\,db + \int_{1}^{a} 0\,db$

$= 0 + \left[\frac{b^2}{2}\right]_0^1 + \left[\frac{b}{2}\right]_1^2 + 0 = \frac{1}{2} + 1 - \frac{1}{2} = 1.$

$$\Rightarrow F_x(a) = \begin{cases} 0 & \text{if } a<0 \\ a^2/2 & \text{if } 0 \le a \le 1 \\ a/2 & \text{if } 1 < a \le 2 \\ 1 & \text{if } a>2 \end{cases}$$



b. Let $b = F_x(a)$

If $0 \le b \le \frac{1}{2}$: $b = \frac{a^2}{2} \Leftrightarrow a^2 = 2b \Leftrightarrow a = \sqrt{2b}$

If $\frac{1}{2} \le b \le 1$: $b = \frac{a}{2} \Leftrightarrow a = 2b$

$$\Rightarrow F_x^{-1}(b) = \begin{cases} \sqrt{2b} & \text{if } 0 \le b \le \frac{1}{2} \\ 2b & \text{if } \frac{1}{2} \le b \le 1. \end{cases}$$

$\Rightarrow$ RV generator: $X = F_x^{-1}(U) = \begin{cases} \sqrt{2U} & \text{if } 0 \le U \le \frac{1}{2} \\ 2U & \text{if } \frac{1}{2} \le U \le 1. \end{cases}$