# Lesson 5. Random Variate Generation

## 0 Warm up

**Example 1.** Let $U$ be a uniformly distributed random variable on $[0, 1]$ (i.e. $U \sim \text{Uniform}[0, 1]$).

a. What is the pdf of $U$?
b. What is the cdf of $U$?

## 1 Overview

- A **random variate** is a particular outcome of a random variable
- Given the cdf of a random variable, how can we generate random variates?
- One method: **the inverse transform method**
- Big picture:
  - We want to generate random variates of $X$ with cdf $F_X$
  - Assume we have a magic box that can generate random variates of $U \sim \text{Uniform}[0, 1]$
  - We will transform random variates from this magic box into random variates of $X$
- How do we do this transformation? Need to define $X$ as a function of $U$

## 2 The discrete case

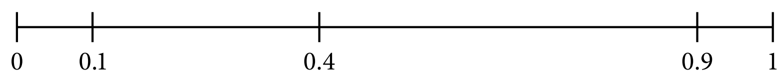
**First, an example**

- Consider the random variable $X$ from the baseball player trade example, with cdf

$$F_X(a) = \begin{cases} 0 & \text{if } a < 1, \\ 0.1 & \text{if } 1 \le a < 2, \\ 0.4 & \text{if } 2 \le a < 3, \\ 0.9 & \text{if } 3 \le a < 4, \\ 1 & \text{if } a \ge 4. \end{cases}$$

- Quick check: what is $p_X(2)$?

- Idea:
  - Assign values of $X$ to values of $U$ (i.e. intervals on $[0, 1]$) according to the cdf




  - Mathematically speaking: set

- Does this transformation work? Let's check for $X = 2$:

- This also works for $X = 1$, $X = 3$, and $X = 4$

**More generally...**

- Let $X$ be a discrete random variable taking values $a_1 < a_2 < a_3 < \dots$

- Define $a_0 = -\infty$ so that $F_X(a_0) =$ [ ]

- A **random variate generator** for $X$ is

$$X = a_i \quad \text{if } F_X(a_{i-1}) < U \leq F_X(a_i) \qquad \text{for } i = 1, 2, \dots$$

- This works because:

[ ]

- To generate a random variate $X$ with cdf $F_X$:

  1: Generate random variate $u$ of $U \sim \text{Uniform}[0,1]$
  2: Find $a_i$ such that $F_X(a_{i-1}) < u \leq F_X(a_i)$
  3: Set $x \leftarrow a_i$
  4: Output $x$, random variate of $X$

- Note: in Nelson, the magic box that generates random variates of $U \sim \text{Uniform}[0,1]$ is represented by the function `random()`
  - In other words, step 1 above can be written as: "Set $u \leftarrow$ `random()`"

## 3 The continuous case

- Now suppose $X$ is a continuous random variable
- We can't assign values of $X$ to intervals of $[0,1]$ – $X$ takes on a continuum of values!
- New, related idea: set $X = F_X^{-1}(U)$
- Why does this transformation work?

- Therefore, $X = F_X^{-1}(U)$ is a **random variate generator** for $X$
- To generate a random variate of $X$ with cdf $F_X$:

  1: Generate random variate $u$ of $U \sim \text{Uniform}[0,1]$
  2: Set $x \leftarrow F_X^{-1}(u)$
  3: Output $x$, random variate of $X$

**Example 2.** Let $X$ be an exponential random variable with parameter $\lambda$. The cdf of $X$ is

$$F_X(a) = \begin{cases} 1 - e^{-\lambda a} & \text{if } a \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Find a random variate generator for $X$.

**Example 3.** Let $X$ be a random variable with pdf

$$f_X(a) = \begin{cases} 0 & \text{if } x < 0 \\ a & \text{if } 0 \le a \le 1 \\ \frac{1}{2} & \text{if } 1 < a \le 2 \\ 0 & \text{if } a > 2 \end{cases}$$

a. Find the cdf of $X$.
b. Find a random variate generator for $X$.