SA402 – Dynamic and Stochastic Models Fall 2013
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# Lesson 9. The Poisson Arrival Process

## 1 Last time...

- **Renewal arrival counting process** (or **renewal process** for short)
  - Interarrival times are independent and time stationary with common cdf $F_G$
  - State variable:

    $$S_n = \text{total number of arrivals up to and including the time of the } n\text{th system event}$$

  - System events:

    $e_0()$: (initialization)

    | | |
    |---|---|
    | 1: $S_n \leftarrow 0$ | (no arrivals initially) |
    | 2: $C_1 \leftarrow F_G^{-1}(\texttt{random()})$ | (set clock for first arrival) |

    $e_1()$: (arrival)

    | | |
    |---|---|
    | 1: $S_{n+1} \leftarrow S_n + 1$ | (one more arrival) |
    | 2: $C_1 \leftarrow T_{n+1} + F_G^{-1}(\texttt{random()})$ | (set clock for next arrival) |

  - Simplified simulation algorithm:

    algorithm Simulation:

    | | |
    |---|---|
    | 1: $n \leftarrow 0$ | (initialize system event counter) |
    | $T_0 \leftarrow 0$ | (initialize event epoch) |
    | $e_0()$ | (execute initial system event) |
    | 2: $T_{n+1} \leftarrow C_1$ | (advance time to next pending system event) |
    | $e_1()$ | (execute system event) |
    | $n \leftarrow n + 1$ | (update event counter) |
    | 3: go to line 2 | |

- Events $\longleftrightarrow$ arrivals: $S_n = n$ for all $n = 0, 1, 2, \ldots$
- Event epoch process: $T_n$ is the time of the $n$th arrival
- Output process: $Y_t = S_n$ for all $t \in [T_n, T_{n+1})$ is the total number of arrivals up to and including time $t$
- The **forward-recurrence time** $R_a$ is the time that passes until the first arrival after time $a$
- From simulation experiments for the Beehunter case, when $G \sim \text{Exponential}(\lambda = 1)$, it looked like $R_6 \sim \text{Exponential}(\lambda = 1)$ as well
  - If this is true, it doesn't matter when we start observing the number of arrivals/accidents
- Today: let's look at the renewal process with $G \sim \text{Exponential}(\lambda)$
  – known as the **Poisson arrival process**, or **Poisson process** for short

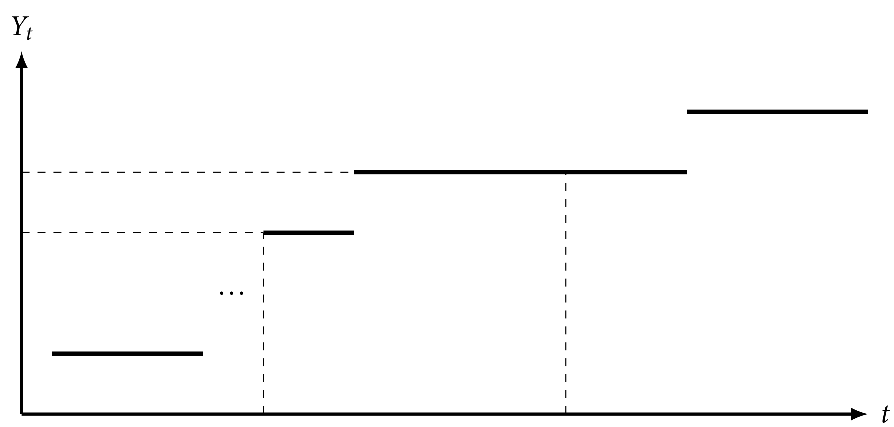
## 2 The Poisson arrival process

- Let $G_i$ be the interarrival time between arrivals $i-1$ and $i$
- We can directly write the event epoch $T_n$ as a function of the interarrival times $G_1, \dots, G_n$:

- Since $F_G$ is the exponential distribution with parameter $\lambda$, $F_{T_n}$ is the Erlang distribution with parameter $\lambda$ and $n$ phases:

$$F_{T_n}(a) = 1 - \sum_{j=0}^{n-1} \frac{e^{-\lambda a}(\lambda a)^j}{j!}$$

- The output process $Y_1, Y_2, \dots$ and the event epoch process $T_1, T_2, \dots$ are fundamentally related:




- Therefore, we can get an explicit expression for the cdf of $Y_t$:

- And we can also get an expression for the pmf of $Y_t$:

- The pmf and cdf may look familiar: $Y_t$ is a **Poisson random variable** with parameter $\lambda t$

$$\Rightarrow \quad E[Y_t] = \lambda t \qquad \text{Var}(Y_t) = \lambda t$$

**Example 1.** In the Beehunter case, the inter-accident times were exponentially distributed with parameter $\lambda = 1$. What is the probability that the total number of accidents at week 24 is greater than 30? (Your calculator can evaluate summations with many terms: use the button.)

## 3 Properties of the Poisson process

- Let $\Delta t > 0$ be a time increment
- The **independent-increments property**: the number of arrivals in nonoverlapping time intervals are independent random variables:
  - As a consequence:
- The **stationary-increments property**: the number of arrivals in a time increment of length $\Delta t$ only depends on the length of the increment, not when it starts:
  - As a consequence:

  $\Rightarrow$ $\lambda$ can be interpreted as the **arrival rate** of the Poisson process
- The **memoryless property**: the forward-recurrence time $R_t$ has the same distribution as the interarrival time:
- These properties make computing probability statements about sample paths of Poisson processes pretty easy

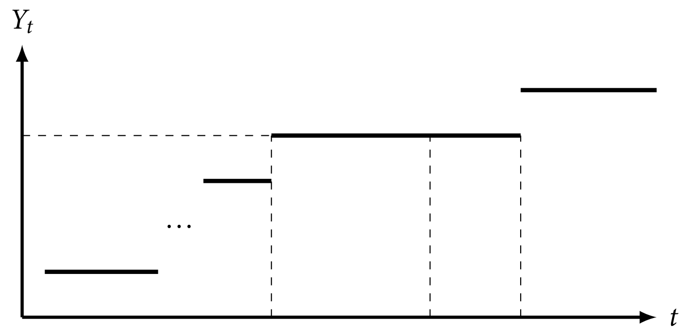
**Example 2.** Recall that in the Beehunter case, a total of 103 accidents have occurred at the intersection up to the time the traffic engineer starts observing, time $a$. What is the probability that more than 30 accidents are observed in the following 24 weeks?

**Example 3.** What is the probability there are 4 accidents at week 5, given that there are 2 accidents at week 4?

- **Any arrival-counting process in which arrivals occur one-at-a-time and has independent and stationary increments must be a Poisson process**
  - If you can justify your system having independent and stationary increments, then you can assume that interarrival times are exponentially distributed
  - This is a very deep powerful result

## 4 Why does the memoryless property hold?

- The memoryless property allows us to ignore when we start observing the Poisson process, since forward-recurrence times and interarrival times are distributed in the same way
- "Memoryless" $\longleftrightarrow$ how much time has passed doesn't matter
- Why is this true for Poisson processes?




$$
\begin{aligned}
\Pr\{R_t > \Delta t \,|\, Y_t = k\} &= \Pr\{R_t > \Delta t \,|\, T_k \le t < T_{k+1}\} \\
&= \Pr\{T_{k+1} - t > \Delta t \,|\, T_k \le t < T_{k+1}\} \\
&= \Pr\{T_k + G_{k+1} - t > \Delta t \,|\, T_k \le t < T_k + G_{k+1}\} \\
&= \Pr\{G_{k+1} > t - T_k + \Delta t \,|\, G_{k+1} > t - T_k, t - T_k > 0\} \\
&= \Pr\{G_{k+1} > t - T_k + \Delta t \,|\, G_{k+1} > t - T_k\} \\
&= \frac{\Pr\{G_{k+1} > t - T_k + \Delta t, G_{k+1} > t - T_k\}}{\Pr\{G_{k+1} > t - T_k\}} \\
&= \frac{\Pr\{G_{k+1} > t - T_k + \Delta t\}}{\Pr\{G_{k+1} > t - T_k\}} \\
&= \frac{e^{-\lambda(t - T_k + \Delta t)}}{e^{-\lambda(t - T_k)}} \\
&= e^{-\lambda \Delta t}
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow \qquad \Pr\{R_t > \Delta t\} &= \sum_{k=0}^{\infty} \Pr\{R_t > \Delta t \,|\, Y_t = k\} \Pr\{Y_t = k\} \\
&= \sum_{k=0}^{\infty} e^{-\lambda \Delta t} \Pr\{Y_t = k\} \\
&= e^{-\lambda \Delta t} \sum_{k=0}^{\infty} \Pr\{Y_t = k\} \\
&= e^{-\lambda \Delta t}
\end{aligned}
$$

- Therefore, $\Pr\{R_t \le \Delta t\} = 1 - e^{-\lambda \Delta t}$, and so $R_t \sim \text{Exp}(\lambda)$
- Along the way, we also showed that $R_t$ and $Y_t$ are independent

- Note: This proof is a little sketchy – we actually need to condition on $T_k$ instead of treating it as a constant
    - Works the same way, but with messier conditional statements and another use of the law of total probability
- The independent-increments and stationary-increments properties follow from the memoryless property and the fundamental relationship between $Y_t$ and $T_n$ (see Nelson pp. 110-111)