SA402 – Dynamic and Stochastic Models | Fall 2013
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Lesson 10. Poisson Processes – Decomposition and Superposition

0 Warm up

Example 1. A radioactive source emits particles according to a Poisson process with interarrival times (in minutes) distributed exponentially with parameter $\lambda = 2$.

a. What is the probability that the first particle appears some time after 3 minutes but before 5 minutes?
b. What is the probability that exactly one particle is emitted in the interval from 3 to 5 minutes?

a. $\Pr\{3 < T_1 < 5\} = \Pr\{T_1 < 5\} - \Pr\{T_1 < 3\} = \left(1 - e^{-2(5)}\right) - \left(1 - e^{-2(3)}\right)$

≈ 0.0024

T_1: $\text{Erlang}(\lambda = 2, n = 1)$
$\parallel$
$\text{Exp}(\lambda = 2)$

b. $\Pr\{Y_5 - Y_3 = 1\} = \Pr\{Y_2 = 1\} = \dfrac{e^{-4}(4)^1}{1!} \approx 0.073$

stationary increments property

Y_2: $\text{Poisson}(\lambda t = 4)$

1 Overview

- Last time: a **Poisson process** is a renewal arrival counting process with interarrival times ~ Exponential(λ)
  - $\Rightarrow$ Expected time between arrivals = $1/\lambda$
  - We say that the Poisson process has an **arrival rate** λ
  - $T_n \sim \text{Erlang}(\lambda, n)$
  - $Y_t \sim \text{Poisson}(\lambda t)$
  - Properties: independent-increments, stationary-increments, memoryless
- Today:
  - When is the Poisson process a good model?
  - Decomposing a Poisson process into two arrival counting subprocesses
  - Superposing (combining) two Poisson processes into one arrival counting process

2 When is the Poisson process a good model?

- Last time: any arrival-counting process in which arrivals occur one-at-a-time and has independent and stationary increments must be a Poisson process
- Independent increments ⇔ number of arrivals in nonoverlapping intervals of time are independent
  - Reasonable when the arrival-counting process is formed by a large number of customers making individual, independent decisions about when to arrive
  - e.g. arrival of telephone calls to a cellular tower
- Stationary increments ⇔ expected number of arrivals = constant rate × length of time interval
  - Reasonable when arrival rate is approximately constant over time
  - e.g. arrivals of cars at a toll booth during evening rush hour

3 Decomposition of Poisson processes

- Let's think back to the Darker Image case:
  - Two types of customers: full-service and self-service
- Suppose that:
  - All customers arrive at the copy shop according to a Poisson process with arrival rate $\lambda = 1/6$
  - 40% of these customers are full-service, 60% self-service
- Let's consider the following model of the arrival process
- Let's model the customer type as a **Bernoulli process** $\{B_1, B_2, \dots\}$ with success probability $\gamma = 0.4$:

$$B_n = \begin{cases} 0 & \text{if } n\text{th customer is self-service -- with probability } 1-\gamma = 0.6 \\ 1 & \text{if } n\text{th customer is full-service -- with probability } \gamma = 0.4 \end{cases}$$

  - In other words, B_n is a Bernoulli random variable with success probability $\gamma = 0.4$
  - $B_1, B_2, \dots$ are independent (and time-stationary)
- Let's assume:
  - $B_1, B_2, \dots$ are also independent of the interarrival times $G_1, G_2, \dots$ with common cdf F_G
  - The common cdf of $B_1, B_2, \dots$ is F_B
- State variables:

$$\begin{aligned} S_n &= \text{total number of customers right after } n\text{th system event} \\ S_{0,n} &= \text{total number of self-service customers right after } n\text{th system event} \\ S_{1,n} &= \text{total number of full-service customers right after } n\text{th system event} \end{aligned}$$

- System events:

$e_0()$: (initialization)

$S_0 \leftarrow 0$	(no customers at start)
$S_{0,0} \leftarrow 0$	(no self-service customers at start)
$S_{1,0} \leftarrow 0$	(no full-service customers at start)
$C_1 \leftarrow F_G^{-1}(\texttt{random()})$	(set clock for first arrival)

$e_1()$: (customer arrival)

$S_{n+1} \leftarrow S_n + 1$	(one more customer)
$B_{n+1} \leftarrow F_B^{-1}(\texttt{random()})$	(determine customer type)
if $\{B_{n+1} = 1\}$ **then**	
$S_{1,n+1} \leftarrow S_{1,n} + 1$	(one more full-service customer)
else	
$S_{0,n+1} \leftarrow S_{0,n} + 1$	(one more self-service customer)
end if	
$C_1 \leftarrow T_{n+1} + F_G^{-1}(\texttt{random()})$	(set clock for next arrival)

- Output process:

$$\mathbf{Y}_t = \begin{pmatrix} Y_t \\ Y_{0,t} \\ Y_{1,t} \end{pmatrix} = \text{number of } \left\{ \begin{array}{l} \text{customers} \\ \text{self-service customers} \\ \text{full-service customers} \end{array} \right\} \text{ up to and including time } t$$

- $\{Y_t; t \geq 0\}$ is a Poisson process with arrival rate λ by construction
- What about $\{Y_{0,t}; t \geq 0\}$ and $\{Y_{1,t}; t \geq 0\}$?
- The **decomposition property**: $\{Y_{0,t} : t \geq 0\}$ and $\{Y_{1,t} : t \geq 0\}$ are <u>independent Poisson subprocesses</u> with arrival rates $\lambda_0 = (1 - \gamma)\lambda$ and $\lambda_1 = \gamma\lambda$ respectively
- This works because the Poisson process is decomposed by a (independent) Bernoulli process
- Other methods of decomposition do not necessarily lead to Poisson subprocesses
- Proof on p. 111 of Nelson

Example 2.

a. What is the probability that fewer than 3 self-service customers arrive during any 60-minute period the shop is open?
b. What is the expected number of full-service customers to arrive during any 60-minute period?

a. self-service arrival process is Poisson w/rate $\lambda_0 = 0.6\left(\frac{1}{6}\right) = \frac{1}{10}$

$$\Rightarrow \Pr\{Y_{0,t+60} - Y_{0,t} < 3\} \underset{\substack{\uparrow \\ \text{stationary increments}}}{=} \Pr\{\underbrace{Y_{0,60}}_{\text{Poisson}(\lambda_0 t = 6)} < 3\} = \Pr\{Y_{0,60} \le 2\} = \sum_{j=0}^{2} \frac{e^{-6}(6)^j}{j!} \approx 0.062$$

b. full-service arrival process is Poisson w/rate $\lambda_1 = 0.4\left(\frac{1}{6}\right) = \frac{1}{15}$

$$\Rightarrow E[Y_{1,t+60} - Y_{1,t}] = E[\underbrace{Y_{1,60}}_{\text{Poisson}(\lambda_1 t = 4)}] = 4$$

Example 3. Suppose we know that 12 customers arrived during the last hour. What is the probability that 3 of them were self-service customers?

$$\Pr\{Y_{0,t+60} - Y_{0,t} = 3 \mid Y_{t+60} - Y_t = 12\} = \Pr\{Y_{0,60} = 3 \mid Y_{60} = 12\} \quad \left(\text{stationary increments applies to both events}\right)$$

$$= \frac{\Pr\{Y_{0,60} = 3,\ Y_{60} = 12\}}{\Pr\{Y_{60} = 12\}} = \frac{\Pr\{Y_{0,60} = 3,\ Y_{1,60} = 9\}}{\Pr\{Y_{60} = 12\}}$$

$$= \frac{\Pr\{Y_{0,60} = 3\}\Pr\{Y_{1,60} = 9\}}{\Pr\{Y_{60} = 12\}} \quad \left(\text{Subprocesses are independent}\right)$$

$$= \left[\frac{e^{-\frac{1}{10}\cdot 60}\left(\frac{1}{10}\cdot 60\right)^3}{3!}\right]\left[\frac{e^{-\frac{1}{15}\cdot 60}\left(\frac{1}{15}\cdot 60\right)^9}{9!}\right]\left[\frac{12!}{e^{-\frac{1}{6}\cdot 60}\left(\frac{1}{6}\cdot 60\right)^{12}}\right]$$

$$= \underbrace{\frac{12!}{3!\,9!}(0.6)^3(0.4)^9}_{\Pr\{\text{Binomial}(n=12,\ p=0.6) = 3\}} \approx 0.012$$

4 Superposition of Poisson processes

- We can also do this in reverse
- Suppose that:
  - self-service customers arrive as a Poisson process with arrival rate $\lambda_0 = 1/9$ and interarrival time cdf F_{G_0}
  - full-service customers arrive according to a Poisson process with arrival rate $\lambda_1 = 1/15$ and interarrival time cdf F_{G_1}
- We can model this arrival process as follows
- Let's use the same state variables as before
- System events:

$e_0()$: (initialization)

$S_0 \leftarrow 0$	(no customers at start)
$S_{0,0} \leftarrow 0$	(no self-service customers at start)
$S_{1,0} \leftarrow 0$	(no full-service customers at start)
$C_1 \leftarrow F_{G_1}^{-1}(\texttt{random()})$	(set clock for next full-service arrival)
$C_2 \leftarrow F_{G_0}^{-1}(\texttt{random()})$	(set clock for next self-service arrival)

$e_1()$: (full-service arrival)

$S_{1,n+1} \leftarrow S_{1,n} + 1$	(one more full-service customer)
$S_{n+1} \leftarrow S_n + 1$	(one more customer, period)
$C_1 \leftarrow T_{n+1} + F_{G_1}^{-1}(\texttt{random()})$	(set clock for next full-service arrival)

$e_2()$: (self-service arrival)

$S_{0,n+1} \leftarrow S_{1,n} + 1$	(one more self-service customer)
$S_{n+1} \leftarrow S_n + 1$	(one more customer, period)
$C_2 \leftarrow T_{n+1} + F_{G_0}^{-1}(\texttt{random()})$	(set clock for next self-service arrival)

- Now by construction, $\{Y_{0,t}; t \geq 0\}$ and $\{Y_{1,t}; t \geq 0\}$ are Poisson processes
- What about $\{Y_t; t \geq 0\}$?
- The **superposition property**: two independent Poisson processes with arrival rates λ_0 and λ_1 that are superposed form a Poisson process with arrival rate $\lambda = \lambda_0 + \lambda_1$
- This works because of the two Poisson processes are independent
- Proof on pp. 111-112 of Nelson

Example 4. Customers arrive at the lobby of the Bank of Simplexville at a rate of 10 per hour. The bank also has a separate ATM where customers arrive at a rate of 20 per hour. Suppose we approximate these arrival processes as Poisson processes. What is the probability that the number of customers arriving at both the lobby and the ATM from 8 am to 12 noon will be greater than 150?

$\Rightarrow$ All customers arrive according to a Poisson process w/arrival rate 30 cust./hr.

$\Rightarrow \Pr\{Y_4 > 150\} = 1 - \Pr\{Y_4 \le 150\} \leftarrow Y_4 \sim \text{Poisson}(\lambda t = 30(4) = 120)$

stationary increments

$$= 1 - \sum_{j=0}^{150} \frac{e^{-120}(120)^j}{j!} \approx 0.0036$$