SA402 – Dynamic and Stochastic Models — Fall 2013
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Lesson 12. Poisson Processes – Review

- A **Poisson process** is a renewal arrival counting process
  - Interarrival times $G_n \sim \text{Exponential}(\lambda)$
    - ⇒ Expected time between arrivals = $E[G_n] = 1/\lambda$
    - ⋄ We say that the Poisson process has an **arrival rate** λ
  - $T_n \sim \text{Erlang}(\lambda, n)$
  - $Y_t \sim \text{Poisson}(\lambda t)$

- **Independent increments**: number of arrivals in nonoverlapping time intervals are independent

$$\Pr\{Y_{t+\Delta t} - Y_t = m \mid Y_t = k\} = \Pr\{Y_{t+\Delta t} - Y_t = m\}$$

- **Stationary increments**: number of arrivals in a time interval only depends on the interval length

$$\Pr\{Y_{t+\Delta t} - Y_t = m\} = \Pr\{Y_{\Delta t} = m\}$$

- **Memoryless**: forward-recurrence time has same distribution as interarrival time

$$R_t \sim \text{Exponential}(\lambda)$$

- Any arrival-counting process in which arrivals occur one-at-a-time and has independent and stationary increments must be a Poisson process
  - Independent increments: reasonable when arrivals are independent, large number
  - Stationary increments: reasonable when arrival rate is approximately constant over time

- **Decomposition of Poisson processes**
  - Poisson process with arrival rate λ
  - Arrival n is type $B_n \sim \text{Bernoulli}(\gamma) = \begin{cases} 0 & \text{with probability } 1-\gamma \\ 1 & \text{with probability } \gamma \end{cases}$
    - ⋄ B_n's are independent of each other and G_n's
  - ⇒ Type 0 arrivals form a Poisson process with arrival rate $\lambda_0 = (1-\gamma)\lambda$,
    Type 1 arrivals form a Poisson process with arrival rate $\lambda_1 = \gamma\lambda$

- **Superposition of Poisson processes**
  - Two independent Poisson processes with arrival rates λ_0 and λ_1, respectively
  - Arrivals from both processes together form a Poisson process with arrival rate $\lambda = \lambda_0 + \lambda_1$

Example 1. The Markov Company has a manufacturing cell that processes jobs during a 12-hour shift starting at 6 a.m. and ending at 6 p.m. Jobs leave the cell according to a Poisson process with rate $\lambda = 8$ per hour.

a. If the cell has processed exactly 10 jobs by 8 a.m., what is the probability that the cell will have processed more than 30 jobs by 10 a.m.?
b. What is the probability that the cell will have processed its 50th jobs before 12 p.m.?
c. If the cell has processed at least 40 jobs by 12 p.m., what is the probability that the cell will have processed its 100th job by the end of the shift?
d. What is the total expected waiting time of the first 4 jobs to be processed? (Assume all jobs are available starting at 6 a.m.)

Example 2. You have been asked to conduct a study of the pedestrian crossing near Chick & Ruth's Delly in Downtown Annapolis. Assume the following behavior. Pedestrians approach the crossing at a rate of 6 pedestrians per minute; one-third of them are on the left side, and two-thirds of them are on the right side. Pedestrians wait until the "WALK" signal, at which time all waiting pedestrians cross instantaneously. Suppose the "WALK" signal occurs every 2 minutes.

a. What is the expected number of pedestrians crossing left to right on a given "WALK" signal?
b. What is the probability that at least one pedestrian crosses right to left on any particular signal?

Example 3. Bart arrives at the southbound platform at the Pentagon Metro station at a random time. Blue Line trains and Yellow Line trains arrive at the platform according to independent Poisson processes with arrival rates of 4 and 6 trains per hour, respectively.

a. What is the probability the next train arrives in 15 minutes?
b. Bart is waiting for a Blue Line train, and has already been waiting for 10 minutes. What is his total expected waiting time?

Example 4. (Nelson 5.15) This problem concerns capacity planning for a manufacturing company. The company has two salespersons, John and Louise, who each cover one half of the United States. At the end of each week, the salespersons report their sales to the company, which then manufactures the products that have been ordered.

The company has three products, which it calls A, B and C for simplicity. Each salesperson obtains 10 orders per week, on average, of which approximately 20% are for A, 70% are for B, and 10% are for C. In terms of capacity, it takes 25 person-hours to produce one A, 15 person-hours to produce one B, and 40 person-hours to produce one C.

Help the company do its capacity planning by answering the following questions. You may assume that the arrival of orders to each salesperson can be well approximated as a Poisson process.

a. A Poisson process is time stationary. What is the physical interpretation of "time stationary" in this situation?
b. What is the probability that the total sales for 1 week will be more than 30 products?
c. Capacity can only be changed on a monthly basis. What is the expected number of person-hours the company will need over a 1-month period?
d. What is the probability that Louise will sell more than 5 product Bs on each of 2 consecutive weeks?