Lesson 13. Nonstationary Poisson Processes – Review

- Consider a Poisson process with arrival rate 1:
  - Interarrival times: $G_n \sim \text{Exponential}(1)$
  - Time of the nth arrival: $T_n \sim \text{Erlang}(1, n)$
  - Number of arrivals up to time t: $Y_t \sim \text{Poisson}(t)$
- A **nonstationary Poisson process** is a renewal arrival counting process that transforms a Poisson process with arrival rate 1 in the following way:
  - The arrival rate $\lambda(\tau)$ is the expected number of arrivals per unit time at time τ
  - The **integrated-rate function** is the expected number of arrivals by time τ:

$$\Lambda(\tau) = \int_0^\tau \lambda(a)\, da$$

  - Time of nth arrival: $U_n = \Lambda^{-1}(T_n)$
  - Number of arrivals in $(\tau, \tau + \Delta\tau]$:

$$Z_{\tau+\Delta\tau} - Z_\tau \sim \text{Poisson}\big(\Lambda(\tau + \Delta\tau) - \Lambda(\tau)\big)$$

  $\Rightarrow$ Expected number of arrivals in $(\tau, \tau + \Delta\tau]$:

$$E[Z_{\tau+\Delta\tau} - Z_\tau] = \Lambda(\tau + \Delta\tau) - \Lambda(\tau)$$

- A nonstationary Poisson process satisfies the **independent-increments property**: number of arrivals in nonoverlapping intervals are independent

$$\Pr\{Z_{\tau+\Delta\tau} - Z_\tau = m \mid Z_\tau = k\} = \Pr\{Z_{\tau+\Delta\tau} - Z_\tau = m\} = \frac{e^{-[\Lambda(\tau+\Delta\tau)-\Lambda(\tau)]}[\Lambda(\tau + \Delta\tau) - \Lambda(\tau)]^m}{m!}$$

- Stationary-increments and memoryless properties do not hold
- For proofs, take a look at the end of Lesson 11 (not too hard)
- "Nonstationary" also known as "nonhomogeneous" or "inhomogeneous"

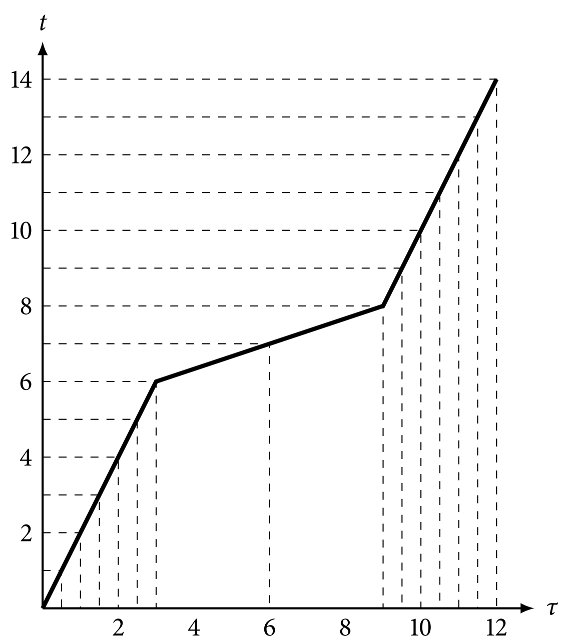
Example 1. Customers arrive at Cantor's Car Repair according to a nonstationary Poisson process with rate

$$\lambda(\tau) = \begin{cases} 2 & \text{if } 0 \le \tau < 3 \\ 1/3 & \text{if } 3 \le \tau < 9 \\ 2 & \text{if } 9 \le \tau < 12 \end{cases}$$

where time is measured in hours, and time 0 is 6 a.m.

a. What is the integrated rate function?
b. What is the inverse of the integrated rate function?

- The graph of $\Lambda(\tau)$ in Example 1 looks like:



- Let τ be the time scale for the nonstationary Poisson process with rate $\lambda(\tau)$
- Let t be the time scale for the stationary Poisson process with rate 1
- The two time scales are connected:

$$t = \Lambda(\tau) \quad \Leftrightarrow \quad \tau = \Lambda^{-1}(t)$$

Example 2. At Cantor's Car Repair, what is the probability that the doctor will see more than 4 customers from 8 a.m to 10 a.m.? What is the expected number of customers in this time period?

Example 3. At Cantor's Car Repair, if 6 customers have arrived by noon, what is the probability that at least 12 customers will have arrived by 6 p.m.?

Example 4. At Cantor's Car Repair, what is the probability that the 10th customer arrives by 4 p.m.?

Example 5. At Cantor's Car Repair, what is the probability that at most 1 customer arrives between 4 p.m and 6 p.m.?

Food for thought

- We defined a nonstationary Poisson process as a transformation from a stationary Poisson process
- What if we wanted to define a nonstationary Poisson process directly as a renewal arrival counting process?
- We would need a distribution for the interarrival times to define the simulation model
- One can derive a conditional cdf for the interarrival times H_n:

$$F_{H_{n+1}|U_n=\tau}(\alpha) = \Pr\{H_{n+1} \leq \alpha \mid U_n = \tau\} = 1 - e^{-[\Lambda(\tau+\alpha)-\Lambda(\tau)]}$$