Lesson 15. Markov Chains – Time-Dependent Performance Measures

0 Warm up – The Case of The Defective Detective

Example 1. Quality control engineers at KRN Corporation are monitoring the performance of a manufacturing system that produces an electronic component. Components are inspected in the sequence they are produced. The engineers believe that there is some dependence between successively produced components, and so they model whether a component is acceptable or defective by a Markov chain with states $\mathcal{M} = \{1, 2\}$ (1 = acceptable, 2 = defective), and one-step transition matrix and initial-state vector

$$\mathbf{P} = \begin{pmatrix} 0.995 & 0.005 \\ 0.495 & 0.505 \end{pmatrix} \qquad \mathbf{p} = \begin{pmatrix} 0.96 \\ 0.04 \end{pmatrix}$$

a. What does $p_{21} = 0.495$ represent?
b. Draw the transition-probability diagram for this Markov chain.
c. What is the probability of the sequence of states $1, 1, 2, 2$?

1 n-step transition probabilities

- Consider a Markov chain with states $\mathcal{M} = \{1, \dots, m\}$
- The n**-step transition probability** $p_{ij}^{(n)}$ from state i to state j:
- Let $\mathbf{P}^{(n)}$ be the $m \times m$ matrix with elements $p_{ij}^{(n)}$
- We can compute n-step transition probabilities using:

Example 2. For the Defective Detective case, what is the probability that the third component is defective, given that the first one is not?

- Why does $\mathbf{P}^{(n)} = \mathbf{P}^n$? Let's look at the proof for $n = 2$:

$$\begin{aligned}
p_{ij}^{(2)} &= \Pr\{S_2 = j \,|\, S_0 = i\} \\
&= \sum_{h=1}^{m} \Pr\{S_2 = j, S_1 = h \,|\,, S_0 = i\} \\
&= \sum_{h=1}^{m} \frac{\Pr\{S_2 = j, S_1 = h, S_0 = i\}}{\Pr\{S_0 = i\}} \\
&= \sum_{h=1}^{m} \frac{p_i p_{ih} p_{hj}}{p_i} \\
&= \sum_{h=1}^{m} p_{ih} p_{hj}
\end{aligned}$$

- The proof for arbitrary n works in a similar fashion
- Also in a similar fashion, we can derive the **Chapman-Kolmogorov equation**:

$$p_{ij}^{(n)} = \sum_{h=1}^{m} p_{ih}^{(k)} p_{hj}^{(n-k)}$$

  - In other words,

$$\begin{aligned}
&\Pr\{\text{going from } i \text{ to } j \text{ in } n \text{ steps}\} \\
&\quad = \sum_{h=1}^{m} \Pr\{\text{going from } i \text{ to } h \text{ in } k \text{ steps, then going from } h \text{ to } j \text{ in } n-k \text{ steps}\}
\end{aligned}$$

2 n-step state probabilities

- The n**-step state probability** $p_j^{(n)}$ for state j is:

- Let $\mathbf{p}^{(n)}$ be the $m \times 1$ vector with elements $p_j^{(n)}$
- We can compute n-step state probabilities using:

Example 3. For the Defective Detective case, what is the probability that the fourth component is defective?

- Why does $\mathbf{p}^{(n)\top} = \mathbf{p}^\top \mathbf{P}^n$?

$$p_j^{(n)} = \Pr\{S_n = j\} = \sum_{i=1}^{m} \Pr\{S_n = j \mid S_0 = i\} \Pr\{S_0 = i\} = \sum_{i=1}^{m} p_i p_{ij}^{(n)}$$

3 First-passage probabilities

- Let $\mathcal{A}$ and $\mathcal{B}$ be two disjoint subsets of the state space $\mathcal{M}$
  - e.g. $\mathcal{M} = \{1, 2, 3, 4\}$, $\mathcal{A} = \{1, 2, 3\}$, $\mathcal{B} = \{4\}$
- Let $m_\mathcal{A}$ and $m_\mathcal{B}$ be the number of states in $\mathcal{A}$ and $\mathcal{B}$, respectively
  - e.g. $m_\mathcal{A} = 3$, $m_\mathcal{B} = 1$
- The **first-passage probability** $f_{ij}^{(n)}$ for initial state $i \in \mathcal{A}$ and final state $j \in \mathcal{B}$ in n time steps is:

  - $f_{ij}^{(n)}$ is the probability that the process is confined to states in $\mathcal{A}$ in the first $n-1$ time steps, but at time step n it enters state $j \in \mathcal{B}$

- Let $\mathbf{P}_{\mathcal{AB}}$ be the submatrix of $\mathbf{P}$ whose elements are p_{ij} with $i \in \mathcal{A}$ and $j \in \mathcal{B}$:
    - e.g. $\mathcal{A} = \{1, 2, 3\}$, $\mathcal{B} = \{4\}$

$$\mathbf{P} = \begin{pmatrix} 0 & 0.95 & 0.01 & 0.04 \\ 0 & 0.27 & 0.63 & 0.10 \\ 0 & 0.36 & 0.40 & 0.24 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad \mathbf{P}_{\mathcal{AB}} = \qquad\qquad \mathbf{P}_{\mathcal{AA}} =$$

- Let $\mathbf{F}^{(n)}_{\mathcal{AB}}$ be the $m_{\mathcal{A}} \times m_{\mathcal{B}}$ matrix whose elements are $f^{(n)}_{ij}$
- We can compute first-passage probabilities using:

Example 4. The Markov chain in the Jungle.com case from Lesson 14 had the following transition probability matrix:

$$\mathbf{P} = \begin{pmatrix} 0 & 0.95 & 0.01 & 0.04 \\ 0 & 0.27 & 0.63 & 0.10 \\ 0 & 0.36 & 0.40 & 0.24 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Recall that a session starts with a log on (state 1) and ends with a log off (state 4) Let X be the length of the session, excluding log on. Compute the probability that $X = 9$.

Example 5. Find the expected session length in the Jungle.com case.

4 Next time...

- What happens in the **long run**, i.e. when the number of time steps n approaches infinity?