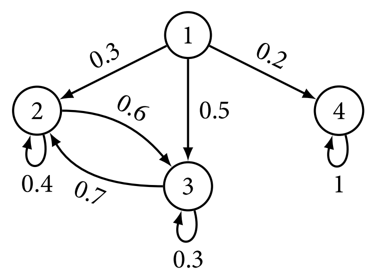
# Lesson 16. Markov Chains – Time-Independent Performance Measures

## 0 Warm up

**Example 1.** An autonomous UAV has been programmed to move between several regions to perform surveillance. The UAV is currently located in region 1, and moves between regions 1, 2, 3, and 4 according to a Markov chain with the following transition-probability diagram:




a. What is the one-step transition probability matrix?
b. What is the probability that the UAV is in region 3 in three time steps?
c. Suppose the UAV is in region 2. What is the probability it is in region 4 in three time steps?

## 1 Overview

- Last time: performance measures that depend on the number of time steps, for example

$$p_{ij}^{(n)} = \Pr\{S_n = j \mid S_0 = i\} \qquad\qquad p_j^{(n)} = \Pr\{S_n = j\}$$

- Today: what happens in the **long run**, i.e. as $n \to \infty$? In particular, what is the **limiting probability**

$$p_{ij}^{(\infty)} = \lim_{n\to\infty} p_{ij}^{(n)} \ \ ?$$

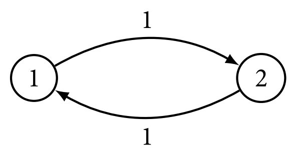
  - Note: the text uses $\vec{p}_{ij}$ instead of $p_{ij}^{(\infty)}$

## 2 Classification of states

- As usual, suppose we have a Markov chain with state space $\mathcal{M} = \{1, \dots, m\}$

### Periodic and aperiodic states

- Consider the following two-state Markov chain:




- The $n$-step transition probability between state 1 and itself is:

- A state $i \in \mathcal{M}$ is **periodic** with period $\delta$ ($\delta$ is a positive integer) if

$$p_{ii}^{(n)} \begin{cases} > 0 & \text{if } n = \delta, 2\delta, 3\delta, \dots \\ = 0 & \text{otherwise} \end{cases}$$

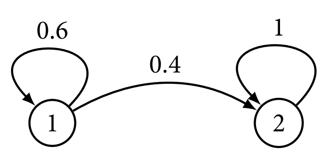
  and therefore $p_{ii}^{(\infty)} = \lim_{n \to \infty} p_{ii}^{(n)}$ does not exist

- A state $i \in \mathcal{M}$ is **aperiodic** if it is not periodic

- In this class, we do not consider $p_{ij}^{(\infty)}$ for periodic states $j$

### Transient and recurrent states

- Consider the following two-state Markov chain:

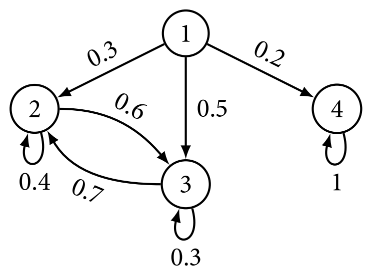



- The limiting probability between state 1 and itself is:

- In other words, the process eventually leaves state 1 and never return

- A state $i \in \mathcal{M}$ is **transient** if $p_{ii}^{(\infty)} = 0$
    - The process will eventually leave state $i$ and never return
- A state $i \in \mathcal{M}$ is **recurrent** if $p_{ii}^{(\infty)} > 0$
    - The process is guaranteed to return to state $i$ over and over again, given that it reaches state $i$ at some time

**Example 2.** Consider the Markov chain in the UAV example.




Can you guess which states are transient and which states are recurrent?

- A subset of states $\mathcal{R}$ **irreducible** if
    - $\mathcal{R}$ forms a self-contained Markov chain
    - no proper subset of $\mathcal{R}$ also forms a Markov chain
- Otherwise the subset $\mathcal{R}$ is **reducible**
- To find transient and recurrent states of a Markov chain:
    1. Find all irreducible proper subsets of the state space
        - If there are no such subsets, the entire state space is irreducible
    2. All states in an irreducible set are recurrent
    3. All states not in an irreducible set are transient

## 3 Performance measures

- Based on the classification of states, we can compute the limiting probabilities $p_{ij}^{(\infty)}$
- **Case 1.** If state $j$ is transient, then
  - state $j$ is transient $\Rightarrow$ will eventually leave state $j$ and never return
- **Case 2.** If states $i$ and $j$ are in different irreducible sets of states, then
  - state $i$ is one self-contained Markov chain, state $j$ is in another
- **Case 3.** If states $i$ and $j$ are in the same irreducible set of states $\mathcal{R}$, then
  - $p_{ij}^{(\infty)} = \pi_j$, for some $\pi_j > 0$, does not depend on $i$
  - Let $\vec{\pi}_{\mathcal{R}}$ be a vector whose elements are $\pi_j$ for $j \in \mathcal{R}$
  - We can compute $\pi_j$ by solving the following system of linear equations:
  - The $\pi_j$ are called **steady-state probabilities**
  - Interpretation: given that the process reaches the irreducible set containing state $j$, $\pi_j$ is
    - the probability of finding the process in state $j$ after a long time, or
    - the long-run fraction of time that the process spends in state $j$
  - Note that by construction, the steady-state probabilities add up to 1

**Example 3.** Consider the UAV example again. Suppose the UAV reaches region 2 at some point. What is the long-run fraction of time that the UAV spends in region 2? Region 3?

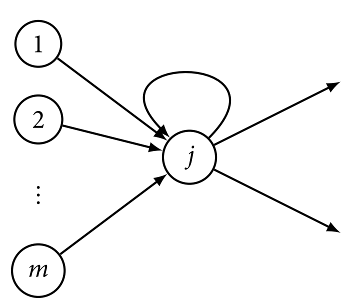
- **Case 4.** If state $i$ is transient and state $j$ is an **absorbing** state (i.e. state $j$ is the only state in an irreducible set of states $\mathcal{R} = \{j\}$), then
  - $p_{ij}^{(\infty)} = \alpha_{ij}$ for some $\alpha_{ij} \geq 0$
  - Let $\mathcal{T}$ be the set of transient states
  - Let $\alpha_{\mathcal{TR}}$ be the vector whose elements are $\alpha_{ij}$ for $i \in \mathcal{T}$ (remember $\mathcal{R} = \{j\}$)
  - We can find the $\alpha_{ij}$'s using:
  - The $\alpha_{ij}$ are called **absorption probabilities**
    - What is the probability that the process is ultimately "absorbed" at state $j$?

**Example 4.** Consider the UAV example again. What is the probability that the UAV is in region 4?

## 4 Why are the steady-state probabilities computed this way?

- Some details in Nelson, pp. 153-154
- Intuition: in steady state, we have that

$$\text{frequency of being in state } j = \text{frequency of transitions into state } j$$



## 5 If we have time... (if not, finish for homework)

**Example 5** (Nelson 6.6b, modified)**.**

a. Classify as recurrent or transient the states of the Markov chain with state space $\{1, 2, 3, 4, 5\}$ and the one-step transition matrix below by first finding all of the irreducible subsets of states.

$$\mathbf{P} = \begin{pmatrix} 0.1 & 0.5 & 0.1 & 0.1 & 0.2 \\ 0.0 & 0.8 & 0.0 & 0.0 & 0.2 \\ 0.0 & 0.0 & 0.3 & 0.0 & 0.7 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.5 & 0.0 & 0.0 & 0.5 \end{pmatrix}$$

b. For each irreducible set of states, find the steady-state probabilities.