SA402 – Dynamic and Stochastic Models — Fall 2013

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Lesson 18. A Quick Start Guide to Markov Processes

1 Overview

- Last few lessons: A Markov chain is a stochastic process that focuses on the state changes, ignoring the actual times at which the changes occur, e.g.
    - Visiting different page types on Jungle.com
    - Movements of a UAV between different regions
- Today: let's look at **Markov processes**, which are similar to Markov chains, but also incorporate the time between state changes
    - We will eventually use these as a framework to study **queueing processes**

2 An algorithmic model of a Markov process

- State space $\mathcal{M} = \{0, 1, 2, \dots, m\}$
    - By convention we include 0
    - For example, the state might represent number of customers in a queue
- State-change process:

$$S_n = n\text{th state visited} \quad \text{for } n = 0, 1, 2, \dots$$

- Initial state probabilities p_j for each $j \in \mathcal{M}$ with cdf F_{S_0}
- **Transition rate** g_{ij} from state i to state j ($i \neq j$)
- **Transition times** $H_{ij} \sim \text{Exponential}(g_{ij})$ from state i to state j ($i \neq j$)
    - All H_{ij}'s are independent of each other
- System events:

$e_i()$: (go to state i, for $i = 0, 1, \dots, m$)

$S_{n+1} \leftarrow i$ (next state is i)
for $j = 0$ to m, $j \neq i$ **do**
  $C_j \leftarrow T_{n+1} + F^{-1}_{H_{ij}}(\texttt{random()})$ (set clocks according to transition rates)
end for

$e_{\text{init}}()$: (initialization)

$S_0 \leftarrow F^{-1}_{S_0}(\texttt{random()})$ (initial state)
for $j = 0$ to m, $j \neq S_0$ **do**
  $C_j \leftarrow T_{n+1} + F^{-1}_{H_{S_0,j}}(\texttt{random()})$ (set clocks according to transition rates)
end for

- The same simulation framework as before (with some minor notation changes):

  algorithm Simulation:

  1: $n \leftarrow 0$ (initialize system event counter)
  $T_0 \leftarrow 0$ (initialize event epoch)
  $e_{\text{init}}()$ (execute initial system event)

  2: $T_{n+1} \leftarrow \min\{C_0, \dots, C_m\}$ (advance time to next pending system event)
  $I \leftarrow \arg\min\{C_0, \dots, C_m\}$ (find index of next system event)

  3: $S_{n+1} \leftarrow S_n$ (temporarily maintain previous state)
  $C_I \leftarrow \infty$ (event I no longer pending)

  4: $e_I()$ (execute system event I)
  $n \leftarrow n + 1$ (update event counter)

  5: go to line 2

- Output process – the state at time t:

$$Y_t = S_n \quad \text{for all } t \in [T_n, T_{n+1})$$

- What's going on here?
  - Suppose the process is in state i
  - The transition time from state i to state j is $H_{ij} \sim \text{Exponential}(g_{ij})$
  - The next state is the one with the lowest transition time

- The transition rates implicitly define the probability of transitioning from one state to the next

- We can draw a **transition rate diagram** for a Markov process, in the same way as the transition probability diagram for a Markov chain, using the transition rates instead of transition probabilities as arc labels

Example 1. Suppose a system has two components that work in series: if one component fails, then the system fails. For each $i \in \{1, 2\}$, the time until component i fails is exponentially distributed with parameter λ_i, and the time to repair component i is also exponentially distributed with parameter μ_i. Model this setting as a Markov process. Draw the transition rate diagram.

3 Time until next transition

- The **overall transition rate** out of state i is $g_{ii} = \sum_{j \neq i} g_{ij}$
- Suppose $S_n = i$ (at time T_n)
- The time of the next event (next transition) is:

$$T_{n+1} = \min_{j=0,\dots,m,j\neq i} \{T_n + H_{ij}\} = T_n + \underbrace{\min_{j=0,\dots,m,j\neq i} \{H_{ij}\}}_{=H_i}$$

- H_i is called the **holding time** in state i
- H_i is the minimum of m independent exponential random variables:

$$H_{ij} \sim \text{Exponential}(g_{ij}) \quad \text{for } j = 0, \dots, m, j \neq i$$

$\Rightarrow$ $H_i \sim \text{Exponential}(\sum_{j\neq i} g_{ij}) = \text{Exponential}(g_{ii})$

- In words, the time until the next transition is exponentially distributed with rate g_{ii}

4 The Markov and time stationarity properties

- The **Markov property**: only the state of the process at the current time t matters for probability statements about future times:

$$\Pr\{Y_{t+\Delta t} = j \mid Y_t = i \text{ and } Y_a \text{ for all } a < t\} = \Pr\{Y_{t+\Delta t} = j \mid Y_t = i\}$$

- The **time-stationarity property**: only the time increment matters, not the starting time:

$$\Pr\{Y_{t+\Delta t} = j \mid Y_t = i\} \text{ is the same for all } t \geq 0$$

- Powerful fact:
  - Let $\{Y_t; t \geq 0\}$ be a continuous-time stochastic process with discrete state space
  - Suppose $\{Y_t; t \geq 0\}$ satisfies the Markov and time stationarity properties

  $\Rightarrow$ $\{Y_t; t \geq 0\}$ must be a Markov process with some transition probabilities g_{ij}

5 Steady state probabilities

- The **steady state probability** of being in state j:

$$\pi_j = \lim_{t \to \infty} \Pr\{Y_t = j\}$$

  - probability of finding the process in state j after a long period of time
  - long-run fraction of time the process is in state j

- How do we compute these probabilities?
  - Over the long run, the transition rate into state j is
  - Over the long run, the transition rate out of state j is
  - These quantities should be equal in steady state

- In matrix form:
  - **G** is the **generator matrix** of the Markov process:

$$\mathbf{G} = \begin{pmatrix} -g_{00} & g_{01} & \cdots & g_{0m} \\ g_{10} & -g_{11} & \cdots & g_{1m} \\ \vdots & & \ddots & \vdots \\ g_{m0} & g_{m1} & \cdots & -g_{mm} \end{pmatrix}$$

  - Then the steady state probabilities can be found by solving

$$\pi^\top \mathbf{G} = \mathbf{0}$$
$$\pi^\top \mathbf{1} = 1$$

Example 2. Find the steady-state probabilities of the Markov process described in Example 1.

Example 3 (Nelson 7.5, modified)**.** The Football State University motor pool maintains a fleet of vans to be used by faculty and students for travel to conferences, field trips, etc. Requests to use a van occur at about 8 per week on average (i.e. 8/7 per day), and a van is used for an average of 2 days. If someone requests a van and one is not available, then the request is denied and other transportation, not provided by the motor pool, must be found. The motor pool currently has 4 vans, but due to university restructuring, it has been asked to reduce its fleet. In order to argue against the proposal, the director of the motor pool would like to predict how many requests for the vans will be denied if the fleet is reduced from 4 to 3.

a. Model the 3-van system as a Markov process.
b. In the long run, what is the rate at which requests are denied?
c. In the long run, what is the average number of vans in use?

6 The Poisson process as a Markov process

- Let $m = \infty$ so that the state space is $\mathcal{M} = \{0, 1, 2, \dots\}$
- Suppose $S_n = n$ (there have been n arrivals so far)
- Recall: interarrival times ~ Exponential(λ)
- The process only transitions into state $n + 1$ at a rate of λ
- The generator matrix **G** looks like:

- What about steady state probabilities?

7 Food for thought

- Why is the minimum of m independent exponential random variables also an exponential random variable?