

## Lesson 19. An Introduction to Queueing Processes

### 0 Warm up

**Example 1.** Let  $Z_i \sim \text{Exponential}(\lambda_i)$  for  $i = 1, \dots, k$  and  $H = \min\{Z_1, \dots, Z_k\}$ . Show that  $H \sim \text{Exponential}(\sum_{i=1}^k \lambda_i)$ .

### 1 The Case of the Last Parking Space on Earth

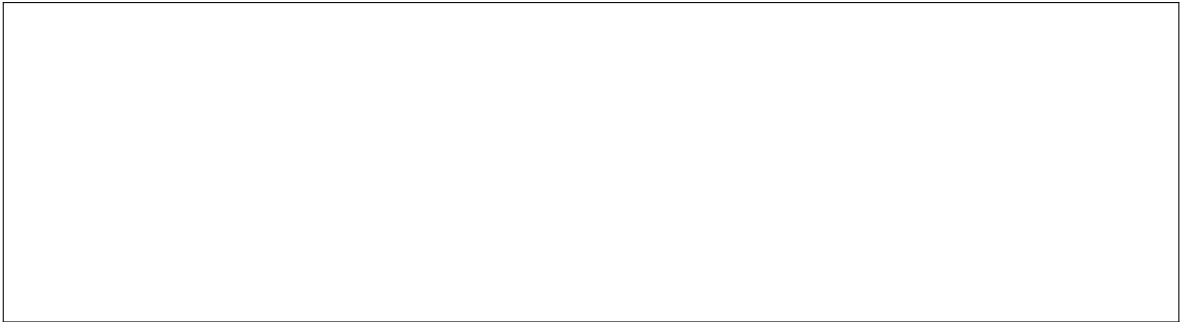
Planning for construction of the proposed “Massive Mall” – the largest shopping mall and indoor golf course in the world – includes determining the amount of customer parking to provide. The developers of Massive Mall have told the planners to “give us enough parking for everybody!” The planners must decide what this is supposed to mean in practice, since the number of parking spaces must be some finite number.

- For the purpose of modeling and analysis, we (the planners) will pretend that parking is unlimited, and then investigate how many spaces are sufficient to satisfy demand a large fraction of the time
- Let’s model this setting as a Markov process
- State space:

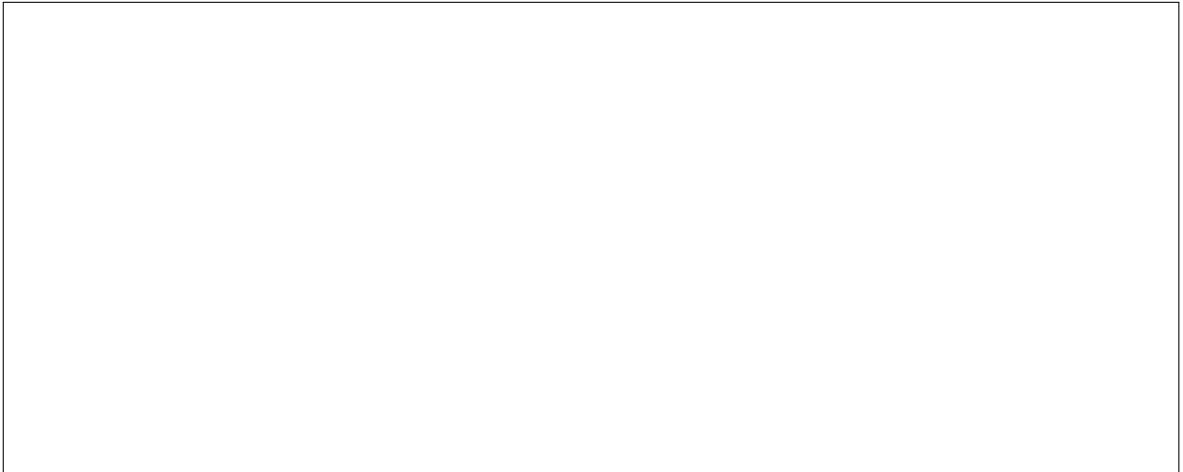
- Input data (random variables):



- Transition rates for car arrivals:



- Transition rates for car departures (parking times):



- Generator matrix:



### 1.1 Performance measures

- Suppose we find the steady-state probabilities  $\pi_0, \pi_1, \pi_2, \dots$
- We might be interested in the long-run expected number of cars in the parking lot:

- We might also be interested in the minimum number of parking spaces  $c^*$  that will accommodate all cars with probability  $1 - \alpha$ :

### 1.2 Is this a reasonable model?

- Car arrivals:
  - Independent?
  - Time-stationary?
  - Exponential interarrival times? (i.e. Poisson arrival process?)
- Car departures (parking times):
  - Independent?
  - Time-stationary?
  - Exponential?

## 2 Next time...

- A generic queueing model: **birth-death processes**
  - Continuous-time Markov process
  - State space  $\mathcal{M} = \{0, 1, \dots\}$  – number of customers in the system (receiving or waiting for service)
  - Generator matrix

$$\mathbf{G} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \dots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & \dots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- transition rate diagram

