Example 1 (Nelson 8.4, modified, cont.). A small ice-cream shop competes with several other ice-cream shops in a busy mall. If there are too many customers already in line at the shop, then potential customers will go elsewhere. Potential customers arrive at a rate of 20 per hour. The probability that a customer will go elsewhere is $j/5$ when there are $j \le 5$ customers already in the system, and 1 when there are $j > 5$ customers already in the system. The server at the shop can serve customers at a rate of 10 per hour. Approximate the process of potential arrivals as Poisson, and the service times as exponentially distributed.

a. Model the process of customer arrivals and departures at this ice-cream shop as a birth-death process (i.e. what are λ_i and μ_i for $i = 0, 1, 2, \dots$?).

b. Over the long run, how many customers are in the shop? (i.e. what is the probability there are 0 customers in the shop? 1? 2? etc.)

c. On average, how many customers are waiting to be served (not including the customer in service)?

d. Over the long run, what fraction of the time is the server busy?

e. What is the effective arrival rate?

f. What is the expected customer delay?

g. What is the expected customer waiting time?

h. Over the long run, at what rate are customers lost?

i. Suppose that the shop makes a revenue of \$2 per customer served and pays the server \$4 per hour. What is the shop's long-run expected profit per hour (revenue minus cost)?

From last time: a. $\lambda_i = \begin{cases} 20(1-\frac{i}{5}) & \text{if } i=0,1,\dots,4 \\ 0 & \text{if } i=5,6,\dots \end{cases}$ $\quad \mu_i = 10 \quad \text{if } i=1,2,\dots$

b.

i	0	1	2	3	4	5	≥ 6
π_i	0.07	0.14	0.22	0.27	0.21	0.09	0

c. $\ell_q = \sum_{j=2}^{\infty} (j-1)\pi_j \approx 1.75$ customers

d. fraction of time busy $= 1 - \pi_0 = 0.93$

e. $\lambda_{eff} = \sum_{j=0}^{\infty} \lambda_j \pi_j = 20\pi_0 + 16\pi_1 + 12\pi_2 + 8\pi_3 + 4\pi_4 = 9.3$ customers/hr

f. $w_q = \frac{\ell_q}{\lambda_{eff}} \approx 0.19$ hrs

g. $w = w_q + \frac{1}{\mu} \approx 0.29$ hrs

h. When there are i customers in the system, $i/5$ of the arriving customers are lost.

$\Rightarrow$ customer loss rate $= 0\pi_0 + 4\pi_1 + 8\pi_2 + 12\pi_3 + 16\pi_4 + 20\pi_5$

≈ 10.7 customers/hr

(Note that the rates in parts e+h add up to 20. Why does this make sense?)

i. expected profit/hr = (expected #customers/hr)(revenue per customer) − cost/hr

$= \lambda_{eff} \cdot 2 - 4 = \$14.6$