Names: Solutions

SA402 – Dynamic and Stochastic Models
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Fall 2013

Lesson 24. Lab – How To Win At Monopoly

Instructions. You have the entire class period to complete this lab. You must work in teams of 3 or 4. Submit only 1 completed lab per team.

If you don't know the rules of Monopoly, here's a very, very basic guide:

- There are 40 board positions, some of which correspond to properties (see the table on page 4).
- All players start at position 1 ("Go"). At each turn, a player rolls 2 six-sided dice and moves according to the sum of their values.
- If the player lands on a property that is not owned by another player, he or she has the opportunity to purchase it. If the property is already owned by another player, then he or she has to pay a fee.
- If the player lands on "Chance" or "Community Chest," then the player must randomly draw a "Chance" or "Community Chest" card, which tells the player to collect money, pay money, or go to a different board position.

If we know which properties are landed on the most, this could help us devise a good strategy on which properties to buy. We will determine which properties are landed on the most by modeling a player's movement as a Markov chain, in which each board position corresponds to a state.

For the Monopoly veterans: we're going to ignore any "rolling doubles" rules (especially the one where 3 double rolls sends a player to Jail). In addition, we're going to assume that a player leaves Jail after 1 turn (that is, going to Jail just moves the player's position). It turns out that these assumptions affect the accuracy of our results only slightly.

1. Download `transition.m` from the course website.

2. The function `transition()` outputs the transition probability matrix **P** for this Monopoly Markov chain:

```
P = transition()
```

3. Let's first try to understand what some of these transition probabilities are. Consider the transition probability between Virginia Avenue (15) and Tennessee Avenue (19).

```
P(15, 19)
```

According to the transition probability matrix **P**, what is its value? Briefly explain why.

$$p_{15,19} = \Pr\{\text{roll } 4\} = \frac{3}{36} \approx 0.0833$$

4. Now consider the transition probability between New York Avenue (20) and Jail (31). According to the transition probability matrix, what is its value? Briefly explain why.

   Hint. You can go to Jail directly from New York Avenue by rolling 11. What happens when you go to the Chance #2 board position (23)? Use the Chance card distribution table on page 4.

$$p_{20,31} = \Pr\{\text{roll } 11\} + \Pr\{\underbrace{\text{roll } 3}_{\text{land on chance}}\} \Pr\{\text{go to jail} \mid \text{land on chance}\}$$

$$= \frac{2}{36} + \frac{2}{36}\left(\frac{1}{16}\right) = \frac{17}{288} \approx 0.0590$$

5. Compute $\mathbf{P}^{1000}$ (no need to write it down here). From this, give an educated guess on which states are transient and which states are recurrent (take a look at your notes if you don't remember what these mean).

$p_{ii}^{1000} > 0$ for all states $i = 1, \ldots, 40$

$\Rightarrow$ educated guess: $\lim_{n\to\infty} p_{ii}^{n} > 0$ for all $i = 1, \ldots, 40$

$\Rightarrow$ <u>all states</u> are recurrent (a state is transient if $\lim_{n\to\infty} p_{ii}^{n} = 0$)

6. Compute the steady-state probabilities by solving the following system of equations:

$$(\mathbf{I} - \mathbf{P})^{\top} \pi = \mathbf{0}$$
$$\mathbf{1}^{\top} \pi = 1$$

   (These equations are the same as the steady-state equations we had before, after some manipulation.)

   Hint. Form the following matrices:

$$\mathbf{A} = \begin{bmatrix} (\mathbf{I} - \mathbf{P})^{\top} \\ 1 \ \cdots \ 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

   Then use the `linsolve` function:

```
pi = linsolve(A, B)
```

   The solution you get should look familiar from question 5. Briefly explain why.

The steady-state probabilities are equal to the limiting probabilities $\lim_{n\to\infty} p_{ii}^{n}$ $(i = 1, \ldots, 40)$ for each irreducible set of states. In this case, there is only 1 irreducible set of states: all states.

7. In the long run, what are the 5 most visited board positions? *Hint.* You can use the `sort` function:

```
[pi_sorted sort_order] = sort(pi)
```

31 - Jail

25 - Illinois Avenue

1 - Go

20 - New York Avenue

19 - Tennessee Avenue

8. (Bonus) Let's consider incorporating the "rolling doubles" rules. In particular:

   - If the player rolls doubles, he or she rolls again after taking his or her turn. If he or she rolls three doubles in a row, then he or she goes to Jail.
   - A player stays in Jail for three turns, or until he or she rolls a double.

   How would you change the Markov chain to incorporate these rules?

Expand the state space: (x,y) for $x=1,\ldots,40$ and $y=0,1,2$.

board position

consecutive doubles rolled so far

Transition probabilities would be computed in a similar fashion.

Board positions

State	Board Position	State	Board Position
1	Go	21	Free Parking
2	Mediterranean Avenue	22	Kentucky Avenue
3	Community Chest #1	23	Chance #2
4	Baltic Avenue	24	Indiana Avenue
5	Income Tax	25	Illinois Avenue
6	Reading Railroad	26	B&O Railroad
7	Oriental Avenue	27	Atlantic Avenue
8	Chance #1	28	Ventnor Avenue
9	Vermont Avenue	29	Water Works
10	Connecticut Avenue	30	Marvin Gardens
11	Jail (we use this as visiting Jail)	31	Go to Jail (we use this as being in Jail)
12	St. Charles Place	32	Pacific Avenue
13	Electric Company	33	North Carolina Avenue
14	States Avenue	34	Community Chest #3
15	Virginia Avenue	35	Pennsylvania Avenue
16	Pennsylvania Railroad	36	Short Line
17	St. James Place	37	Chance #3
18	Community Chest #2	38	Park Place
19	Tennessee Avenue	39	Luxury Tax
20	New York Avenue	40	Boardwalk

Chance card distribution

Destination	Probability
Go (1)	1/16
Reading Railroad (6)	1/16
St. Charles Place (12)	1/16
Illinois Avenue (25)	1/16
Jail (31)	1/16
Boardwalk (40)	1/16
Nearest utility (forward direction)	1/16
Nearest railroad (forward direction)	1/16
3 spaces back	1/16
Stay put	7/16

Community Chest card distribution

Destination	Probability
Go (1)	1/16
Jail (31)	1/16
Stay put	14/16