Solutions to Problem 1.

Note that the cdf F_X only changes value at 2, 4, 5. Therefore, X only takes values 2, 4, and 5 with positive probability.

Why is this true? Consider $X = 4$. Roughly speaking,

$$\Pr\{X = 4\} = \Pr\{4 - \varepsilon < X \leq 4\}.$$

for some very small positive value of ε.[1] Therefore,

$$\Pr\{X = 4\} = \Pr\{4 - \varepsilon < X \leq 4\} = F_X(4) - F_X(4 - \varepsilon) = 0.5.$$

Note that $F_X(4)$ and $F_X(4 - \varepsilon)$ are different because F_X changes value at 4.

On the other hand, consider $X = 3$. Again, roughly speaking,

$$\Pr\{X = 3\} = \Pr\{3 - \varepsilon < X \leq 3\}$$

and so

$$\Pr\{X = 3\} = \Pr\{3 - \varepsilon < X \leq 3\} = F_X(3) - F_X(3 - \varepsilon) = 0.$$

So, X does not take the value 3 with positive probability. Note that $F_X(3)$ and $F_X(3 - \varepsilon)$ are the same because F_X does not change value at 3.

a. The pmf of X is

$$\begin{aligned} p_X(2) &= \Pr\{X = 2\} = \Pr\{X \leq 2\} = F_X(2) = 0.4 \\ p_x(4) &= \Pr\{X = 4\} = \Pr\{2 < X \leq 4\} = F_X(4) - F_X(2) = 0.5 \\ p_X(5) &= \Pr\{X = 5\} = \Pr\{4 < X \leq 5\} = F_X(5) - F_X(4) = 0.1 \end{aligned}$$

b. The expected value of X is

$$E[X] = 2(0.4) + 4(0.5) + 5(0.1) = 3.3$$

c. The variance of X is

$$\text{Var}(X) = (2 - 3.3)^2(0.4) + (4 - 3.3)^2(0.5) + (5 - 3.3)^2(0.1) = 1.21$$

d. No, Professor Wright is not correct. $F_X(3)$ gives the probability that X is less than or equal to 3, not the probability that X is equal to 3. Furthermore, as we discussed above, $\Pr\{X = 3\} = 0$.

Solutions to Problem 2.

a. The probability that $2 < X \leq 3$ is

$$\Pr\{2 < X \leq 3\} = \int_2^3 f_x(a)\,da = \int_2^3 \left(\frac{1}{4}a - \frac{1}{4}\right) da = \frac{3}{8}$$

b. The expected value of X is

$$E[X] = \int_{-\infty}^{\infty} a f_X(a) da = \int_1^3 a\left(\frac{1}{4}a - \frac{1}{4}\right) da + \int_3^4 a\left(\frac{1}{2}\right) da \approx 2.92$$

c. $\Pr\{X \leq 6\} = 1$, because the maximum value that X can take (with positive probability) is 4 (see part d).

d. No, Professor Wright is not correct. The maximum value that X can take (with positive probability) is 4, because $f_X(a) = 0$ for all $a > 4$.

[1] To be completely correct, $\Pr\{X = 4\} = \lim_{\varepsilon \to 0^+} \Pr\{4 - \varepsilon < 4 \leq 4\}$. Recall that $\lim_{\varepsilon \to 0^+}$ denotes the limit as ε approaches 0 from the right.