SA402 Fall 2022 · Final Exam · List of Useful Formulas

Poisson arrival processes

- $G_n \sim$ Exponential random variable with parameter λ:

$$F_{G_n}(a) = \begin{cases} 1 - e^{-\lambda a} & \text{if } a \geq 0 \\ 0 & \text{if } a < 0 \end{cases} \qquad E[G_n] = \frac{1}{\lambda}$$

- $T_n \sim$ Erlang random variable with parameter λ and n phases:

$$F_{T_n}(a) = \begin{cases} 1 - \sum_{j=0}^{n-1} \frac{e^{-\lambda a}(\lambda a)^j}{j!} & \text{if } a \geq 0 \\ 0 & \text{if } a < 0 \end{cases} \qquad E[T_n] = \frac{n}{\lambda}$$

- $Y_t \sim$ Poisson random variable with parameter λt:

$$p_{Y_t}(n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \quad \text{for } n = 0, 1, 2, \ldots \qquad E[Y_t] = \lambda t$$

Markov chains

- n-step transition probabilities: $\mathbf{P}^{(n)} = \mathbf{P}^n$
- n-step state probabilities: $\mathbf{p}^{(n)\top} = \mathbf{p}^\top \mathbf{P}^n$
- First-passage probabilities, starting in $\mathcal{A}$ and ending in $\mathcal{B}$ in the nth step: $\mathbf{F}^{(n)}_{\mathcal{AB}} = \mathbf{P}^{n-1}_{\mathcal{AA}} \mathbf{P}_{\mathcal{AB}}$
- Steady-state probabilities for irreducible set $\mathcal{R}$:

$$\pi_{\mathcal{R}}^\top \mathbf{P}_{\mathcal{RR}} = \pi_{\mathcal{R}}^\top$$
$$\pi_{\mathcal{R}}^\top \mathbf{1} = 1$$

- Absorption probabilities for transient states $\mathcal{T}$ and absorbing state $\mathcal{R} = \{j\}$: $\alpha_{\mathcal{TR}} = (\mathbf{I} - \mathbf{P}_{\mathcal{TT}})^{-1} \mathbf{P}_{\mathcal{TR}}$

Markov processes

- Steady-state probabilities:

$$\pi^\top \mathbf{G} = \mathbf{0}$$
$$\pi^\top \mathbf{1} = 1$$

Birth-death processes

- Steady-state probabilities:

$$d_0 = 1 \qquad d_j = \frac{\lambda_0 \lambda_1 \cdots \lambda_{j-1}}{\mu_1 \mu_2 \cdots \mu_j} \quad \text{for } j = 1, 2, \ldots \qquad D = \sum_{i=0}^{\infty} d_i \qquad \pi_j = \frac{d_j}{D} \quad \text{for } j = 0, 1, 2, \ldots$$

- Expected number of customers in the system: $\ell = \sum_{n=0}^{\infty} n \pi_n$
- Expected number of customers in the queue, s parallel servers: $\ell_q = \sum_{n=s+1}^{\infty} (n-s)\pi_n$
- Effective arrival rate: $\lambda_{\text{eff}} = \sum_{i=0}^{\infty} \lambda_i \pi_i$
- Little's law (system-wide): $\ell = \lambda_{\text{eff}} w$
- Little's law (queue only): $\ell_q = \lambda_{\text{eff}} w_q$

Standard queueing models

M/M/∞:

- Steady-state probabilities: $\pi_j = \Pr\{L = j\}$ where L is a Poisson random variable with parameter λ/μ

M/M/s:

- Steady-state probabilities:

$$\rho = \frac{\lambda}{s\mu} \qquad \pi_0 = \left[\left(\sum_{j=0}^{s} \frac{(s\rho)^j}{j!} \right) + \frac{s^s \rho^{s+1}}{s!(1-\rho)} \right]^{-1} \qquad \pi_j = \begin{cases} \dfrac{(\lambda/\mu)^j}{j!}\pi_0 & \text{for } j = 1, 2, \ldots, s \\ \dfrac{(\lambda/\mu)^j}{s! s^{j-s}}\pi_0 & \text{for } j = s+1, s+2, \ldots \end{cases}$$

- Expected number of customers in queue: $\ell_q = \dfrac{\pi_s \rho}{(1-\rho)^2}$
- Expected number of customers in the system: $\ell = \ell_q + \dfrac{\lambda}{\mu}$

G/G/s:

- Whitt's approximation:

$$G = \text{generic interarrival time random variable with rate } \lambda = \frac{1}{E[G]}$$

$$X = \text{generic service time random variable with rate } \mu = \frac{1}{E[X]}$$

$$\varepsilon_a = \frac{\text{Var}[G]}{E[G]^2} \qquad \varepsilon_s = \frac{\text{Var}[X]}{E[X]^2} \qquad \hat{w}_q \approx \frac{\varepsilon_a + \varepsilon_s}{2} w_q$$