Lesson 11. Random Variate Generation

0 Warm up

Example 1. Suppose $U \sim \text{Uniform}[0,1]$.

a. What is $\Pr\{U \leq 0.3\}$? $\Pr\{U \leq -0.7\}$? $\Pr\{U \leq 1.5\}$?
b. In general, what is the cdf F_U of U?

1 Overview

- A **random variate** is a particular outcome of a random variable
- How can we generate random variates?
- One method: **the inverse transform method**
- Big picture:
    - We want to generate random variates of X with cdf F_X
    - We have a pseudo-random number generator
        - i.e. pseudo-random numbers, or samples from $U \sim \text{Uniform}[0,1]$
    - We will transform these pseudo-random numbers into random variates of X
- How do we do this transformation? Need to define X as a function of U

2 The discrete case

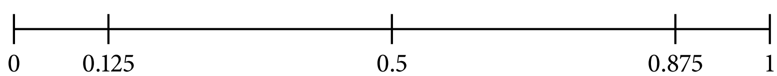
2.1 An example

- Consider a binomial random variable X with 3 trials and success probability 0.5
- X has cdf

$$F_X(a) = \Pr\{X \leq a\} = \sum_{k:k\leq a} \binom{3}{k}\left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{3-k} = \begin{cases} 0 & \text{if } a < 0, \\ 0.125 & \text{if } 0 \leq a < 1, \\ 0.5 & \text{if } 1 \leq a < 2, \\ 0.875 & \text{if } 2 \leq a < 3, \\ 1 & \text{if } a \geq 3. \end{cases}$$

- Quick check: what is $p_X(2) = \Pr\{X = 2\}$? *Hint.* $\Pr\{a < X \leq b\} = \Pr\{X \leq b\} - \Pr\{X \leq a\}$.

- Idea:
  - Assign values of X to values of U (i.e. intervals on $[0,1]$) according to the cdf



  - Mathematically speaking: set

- Does this transformation work? Let's check for $X = 2$:

- This also works for $X = 1$, $X = 3$, and $X = 4$

More generally...

- Let X be a discrete random variable taking values $a_1 < a_2 < a_3 < \dots$
- Define $a_0 = -\infty$ so that $F_X(a_0) = 0$
- A **random variate generator** for X is

$$X = a_i \quad \text{if } F_X(a_{i-1}) < U \leq F_X(a_i) \qquad \text{for } i = 1, 2, \dots$$

- So, to generate a random variate X with cdf F_X:

  Generate pseudo-random number u (i.e. sample from $U \sim \text{Uniform}[0,1]$)
  Find a_i such that $F_X(a_{i-1}) < u \leq F_X(a_i)$
  Set $x \leftarrow a_i$
  Output x, random variate of X

3 The continuous case

- Now suppose X is a continuous random variable
- We can't assign values of X to intervals of $[0,1]$ – X takes on a continuum of values!
- New, related idea: set $X = F_X^{-1}(U)$
- Why does this transformation work?

- Therefore, $X = F_X^{-1}(U)$ is a **random variate generator** for X
- To generate a random variate of X with cdf F_X:

  Generate pseudo-random number u
  Set $x \leftarrow F_X^{-1}(u)$
  Output x, random variate of X

Example 2. Let X be an exponential random variable with parameter λ. The cdf of X is

$$F_X(a) = \begin{cases} 1 - e^{-\lambda a} & \text{if } a \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Find a random variate generator for X.