SA421 – Simulation Modeling Fall 2015
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Lesson 11. Random Variate Generation

1 Overview

- A **random variate** is a particular outcome of a random variable
- How can we generate random variates?
- One method: **the inverse transform method**
- Big picture:
  - We want to generate random variates of X with cdf F_X
  - We have a pseudo-random number generator
    - i.e. pseudo-random numbers, or samples from $U \sim \text{Uniform}[0,1]$
  - We will transform these pseudo-random numbers into random variates of X
- How do we do this transformation?

2 The discrete case

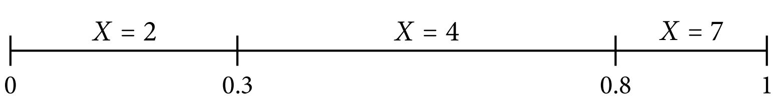
Example 1. Consider the random variable X with cdf

$$F_X(a) = \begin{cases} 0 & \text{if } a < 2, \\ 0.3 & \text{if } 2 \leq a < 4, \\ 0.8 & \text{if } 4 \leq a < 7, \\ 1 & \text{if } a \geq 7. \end{cases}$$

a. What values can X take?

b. What is $p_X(4) = \Pr\{X = 4\}$? *Hint.* $\Pr\{a < X \leq b\} = \Pr\{X \leq b\} - \Pr\{X \leq a\}$.

c. Let's split the interval $[0,1]$ into subintervals according to the cdf F_X, and assign values of X to each subinterval like so:



Suppose you generate a lot of random numbers (samples from Uniform$[0,1]$). What percentage of them will belong to the interval corresponding to $X = 4$?

More generally...

- Let X be a discrete random variable taking values $a_1 < a_2 < a_3 < \dots$
- Define $a_0 = -\infty$ so that $F_X(a_0) = 0$
- A **random variate generator** for X is

$$X = a_i \quad \text{if } F_X(a_{i-1}) < U \leq F_X(a_i) \qquad \text{for } i = 1, 2, \dots \quad \text{where } U \sim \text{Uniform}[0,1]$$

- So, to generate a random variate of X with cdf F_X:

1: Generate pseudo-random number u (i.e. sample from $U \sim \text{Uniform}[0,1]$)
2: Find a_i such that $F_X(a_{i-1}) < u \leq F_X(a_i)$
3: Set $x \leftarrow a_i$
4: Output x, random variate of X

3 The continuous case

- Now suppose X is a continuous random variable
- We can't assign values of X to intervals of $[0,1]$ – X takes on a continuum of values!
- New, related idea: set $X = F_X^{-1}(U)$ where $U \sim \text{Uniform}[0,1]$
- Why does this transformation work?

$$\Pr\{X \leq a\} = \Pr\left\{F_X^{-1}(U) \leq a\right\} = \Pr\left\{F_X\big(F_X^{-1}(U)\big) \leq F_X(a)\right\} = \Pr\left\{U \leq F_X(a)\right\} = F_X(a)$$

- Therefore, $X = F_X^{-1}(U)$ is a **random variate generator** for X
- To generate a random variate of X with cdf F_X:

1: Generate pseudo-random number u
2: Set $x \leftarrow F_X^{-1}(u)$
3: Output x, random variate of X

Example 2.

Let X be a continuous random variable with cdf

$$F_X(a) = \begin{cases} 0 & \text{if } a < 0 \\ a^2 & \text{if } 0 \leq a < 1 \\ 1 & \text{if } a \geq 1 \end{cases}$$

Find a random variate generator for X.