SM223 – Calculus III with Optimization — Fall 2012
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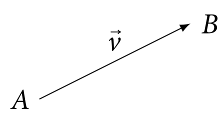
Lesson 2. Vectors

1 Today...

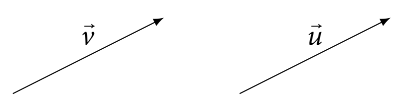
- Vectors graphically
- Vectors algebraically
- Standard basis vectors and unit vectors
- Problems with forces

2 Vectors graphically

- A **vector** is an object that has both __________ and __________
- Often represented by an arrow:
  - Length of arrow represents magnitude of vector
  - Arrow points in direction of vector
- Notation: $\mathbf{v}$ or $\vec{v}$



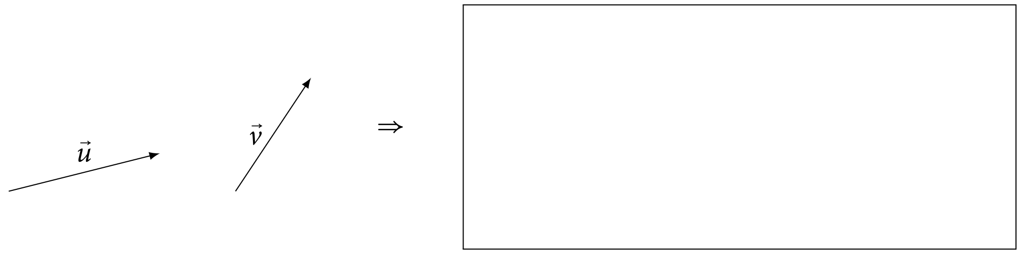
- A vector has an **initial point** (the tail) and a **terminal point** (the tip)
- Two vectors are **equivalent** if they have the same magnitude and direction
  - For example: $\vec{u}$ and $\vec{v}$ are equivalent, <u>even though they are in different positions</u>



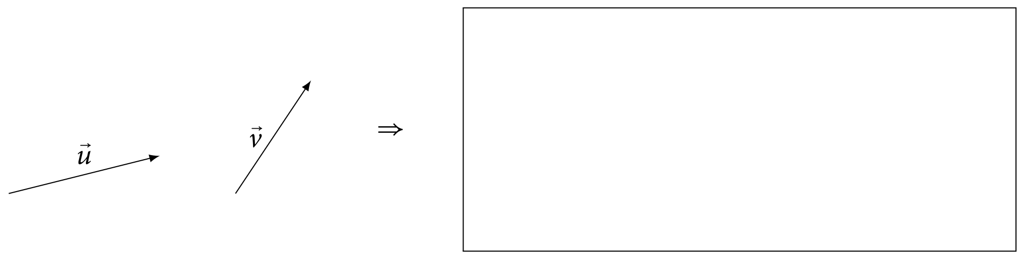
- The **zero vector** $\vec{0}$ has magnitude 0
  - $\vec{0}$ is the only vector with no specific direction

2.1 Adding vectors

- Let $\vec{u}, \vec{v}$ be vectors $\Rightarrow$ $\vec{u} + \vec{v}$ is another vector
- **Triangle law** for adding vectors:



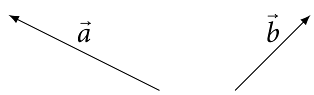
- **Parallelogram law** for adding vectors:



2.2 Scalar multiplication

- Let c be a scalar, $\vec{v}$ be a vector $\Rightarrow$ $c\vec{v}$ is another vector
  - If $c > 0$, then $c\vec{v}$ is a vector in the same direction as $\vec{v}$ and $|c|$ times the length of $\vec{v}$
  - If $c < 0$, then $c\vec{v}$ is a vector in the opposite direction as $\vec{v}$ and $|c|$ times the length of $\vec{v}$
  - If $c = 0$, then $c\vec{v} = \vec{0}$
- Note that we can subtract vectors: $\vec{u} - \vec{v} = \vec{u} + (-1\vec{v})$

Example 1. Consider the vectors drawn below:

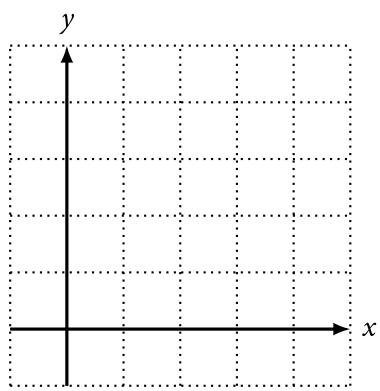


Draw: (a) $2\vec{b}$ (b) $2\vec{b} - \vec{a}$

3 Vectors algebraically

- How do we represent vectors numerically?
- A vector $\vec{a}$ can be represented by an ordered list of numbers:
  - In $\mathbb{R}^2$: $\vec{a} = \langle a_1, a_2 \rangle$
  - In $\mathbb{R}^3$: $\vec{a} = \langle a_1, a_2, a_3 \rangle$
- These numbers (e.g. a_1, a_2, a_3) are known as **components** of $\vec{a}$
  - The components of a vector indicate the distance between the initial point and the terminal point in each coordinate axis direction

Example 2. Draw the following vectors in $\mathbb{R}^2$: (a) $\vec{a} = \langle 1, 2 \rangle$ (b) $\vec{b} = \langle 3, -1 \rangle$



- Let's stick with $\mathbb{R}^3$ for now
- Let $\vec{a}$ be a vector in $\mathbb{R}^3$ that starts at point $A(x_1, y_1, z_1)$ and ends at $B(x_2, y_2, z_2)$

$\Rightarrow$ $\vec{a} =$

- The **magnitude** or **length** of vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ is

- Adding vectors: $\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle =$
- Subtracting vectors: $\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle =$
- Scalar multiplication: $c\langle a_1, a_2, a_3 \rangle =$

Example 3. Let $\vec{a} = \langle 2, -4, 3\rangle$ and $\vec{b} = \langle -4, 0, 2\rangle$.

(a) $|\vec{a}| =$ [ ]

(b) $2\vec{a} - 3\vec{b} =$ [ ]

- **Properties of vectors**
  1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
  2. $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$
  3. $\vec{a} + \vec{0} = \vec{a}$
  4. $\vec{a} + (-\vec{a}) = \vec{0}$
  5. $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$
  6. $(c + d)\vec{a} = c\vec{a} + d\vec{a}$
  7. $(cd)\vec{a} = c(d\vec{a})$
  8. $1\vec{a} = \vec{a}$

4 Standard basis vectors and unit vectors

- **Standard basis vectors** in $\mathbb{R}^3$
  - $\vec{i} =$ [ ]
  - $\vec{j} =$ [ ]
  - $\vec{k} =$ [ ]
- We can write any vector as the sum of scalar multiples of standard basis vectors:

$$\langle a_1, a_2, a_3\rangle = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

- A **unit vector** is a vector with length 1
  - For example, $\vec{i}, \vec{j}, \vec{k}$ are all unit vectors
- If $\vec{a} \neq \vec{0}$, then the unit vector that has the same direction as $\vec{a}$ is $\dfrac{1}{|\vec{a}|}\vec{a}$

Example 4. Let $\vec{a} = 4\vec{i} - \vec{j} + 2\vec{k}$ and $\vec{b} = \vec{i} + 2\vec{k}$.

(a) Write $\vec{a} - 2\vec{b}$ in terms of $\vec{i}, \vec{j}, \vec{k}$.

(b) Find a unit vector in the direction of $\vec{a} - 2\vec{b}$.

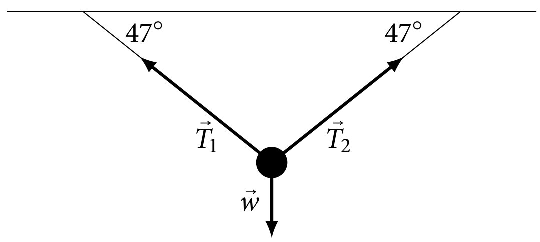
[ ]

- Note: everything we've discussed in Sections 3 and 4 apply to vectors in $\mathbb{R}^2$ in a similar way

5 Problems with forces

- Some physics:
  - **Force** has magnitude and direction, and so it can be represented by a vector
  - Force is measured in pounds (lbs) or newtons (N)
  - If several forces are acting on an object, the **resultant force** experienced by the object is the sum of these forces

Example 5. A weight $\vec{w}$ counterbalances the tensions (forces) in two wires as shown below:



The tensions $\vec{T}_1$ and $\vec{T}_2$ both have a magnitude of 20lb. Find the magnitude of the weight $\vec{w}$.

- Note: if an object has a mass of m kg, then it has a weight of mg N, where $g = 9.8$