SM223 – Calculus III with Optimization — Fall 2012
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Lesson 5. The Cross Product

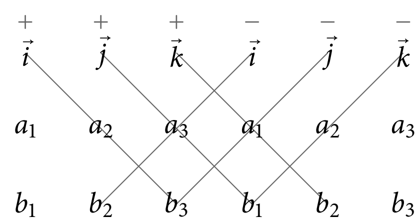
1 Today...

- Definitions and properties
- Applications to torque

2 Definitions and properties

- If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of $\vec{a}$ and $\vec{b}$ is

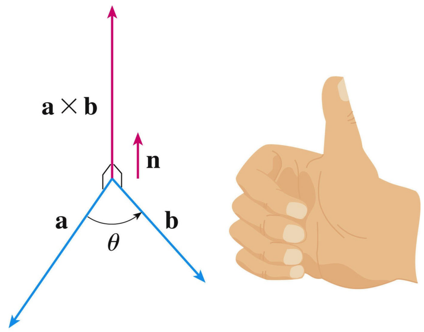
- Note: $\vec{a} \times \vec{b}$ is a vector (unlike the dot product)
- The cross product is only defined for 3D vectors
- Mnemonic for taking the cross product:



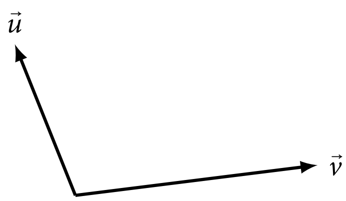
Example 1. Let $\vec{a} = \langle 1, 3, 4 \rangle$ and $\vec{b} = \langle 2, 7, -5 \rangle$. Find $\vec{a} \times \vec{b}$.

- The vector $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.
- Orthogonal which way? **Right-hand rule**
  - Curl fingers of right hand from $\vec{a}$ to $\vec{b}$

  $\Rightarrow$ Thumb points in direction of $\vec{a} \times \vec{b}$



Example 2. Find the direction of $\vec{u} \times \vec{v}$.

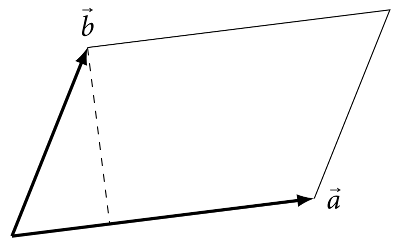


Example 3. Find two unit vectors orthogonal to both $2\vec{j} - \vec{k}$ and $\vec{i} + 4\vec{j}$.

- What about the magnitude of $\vec{a} \times \vec{b}$?
- If θ is the angle between $\vec{a}$ and $\vec{b}$, then
- $\sin\theta = 0$ when $\theta =$

⇒ Two nonzero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if

- $|\vec{a} \times \vec{b}|$ = the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$:

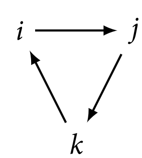


Example 4. Find the area of the triangle with vertices $P(1, 4, 2)$, $Q(-2, 5, -1)$, and $R(1, 3, 1)$.

- Cross products between $\vec{i}$, $\vec{j}$ and $\vec{k}$ are pretty easy to remember:

$$\vec{i} \times \vec{j} = \vec{k} \qquad \vec{j} \times \vec{k} = \vec{i} \qquad \vec{k} \times \vec{i} = \vec{j}$$
$$\vec{j} \times \vec{i} = -\vec{k} \qquad \vec{k} \times \vec{j} = -\vec{i} \qquad \vec{i} \times \vec{k} = -\vec{j}$$

  - Mnemonic:



- **Properties of cross products:** if $\vec{a}, \vec{b}, \vec{c}$ are vectors and c is a scalar:

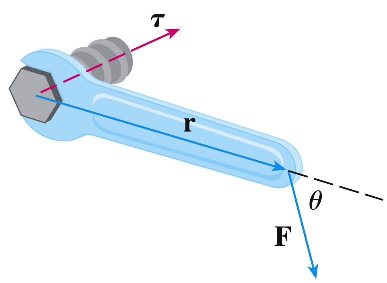
  1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
  2. $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$
  3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
  4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
  5. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
  6. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

- The cross product is not commutative, i.e., $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$

- The cross product is not associative either, i.e. $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

3 Torque

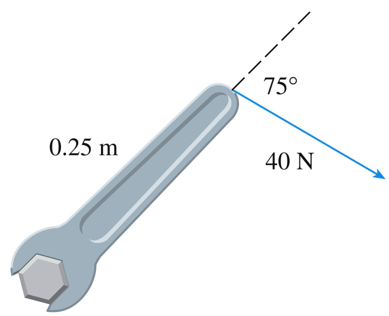
- Suppose we have a force $\vec{F}$ acting on a rigid body at a point given by position vector $\vec{r}$
- For example, tightening a bolt:



- The **torque** (relative to the origin) is

- The magnitude of the torque is

Example 5. A bolt is tightened by applying a 40-N force to a 0.25m wrench as shown below.



Find the magnitude of the torque about the center of the bolt.