SM223 – Calculus III with Optimization — Fall 2012
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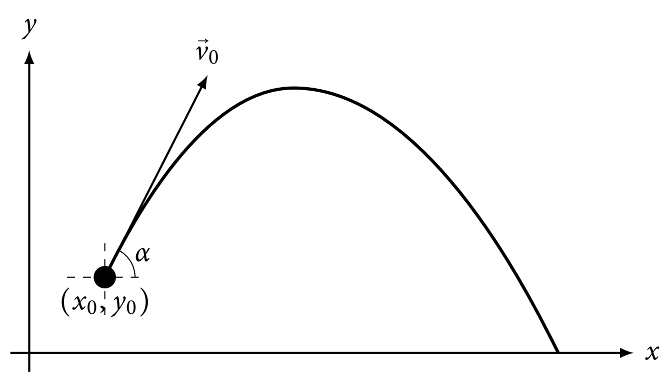
Lesson 15. Projectile Motion

1 Today...

- Trajectory of a projectile
- Horizontal distance traveled by a projectile
- Vertical height reached by a projectile, shape of the trajectory

2 Trajectory of a projectile

- A projectile with mass m is fired
  - initial point (x_0, y_0)
  - angle of elevation α
  - initial velocity $\vec{v}_0$
- Assume:
  - Air resistance is negligible
  - The only external force is due to gravity



- We (you) will derive parametric equations that describe the trajectory of this projectile
- Recall Newton's second law of motion: if at any time t, a force $F(t)$ acts on an object of mass m producing an acceleration $\vec{a}(t)$, then $\vec{F}(t) = m\vec{a}(t)$.

1. Let's define $v_0 = |\vec{v}_0|$ (we're just renaming the initial speed, or the magnitude of the initial velocity). Using this new notation, write $\vec{v}_0$ in terms of v_0 and α. *Hint.* You'll need to use trigonometry.

$\vec{v}_0 = \langle v_0 \cos\alpha, v_0 \sin\alpha \rangle$

2. We need an expression for the acceleration $\vec{a}$ of the projectile. Since the only external force is due to gravity, which acts downward, we have that $\vec{F}(t) = m\vec{a}(t) = \langle 0, -mg \rangle$. Using this, write an expression for $\vec{a}(t)$.

$\vec{a}(t) = \langle 0, -g \rangle$

3. Using your answer from part 2, write an expression for the velocity $\vec{v}(t)$ of the projectile.
Hint 1. Recall that $\vec{a}(t) = \vec{v}'(t)$. *Hint 2.* Don't forget the constant vector of integration. *Hint 3.* Since the initial velocity is $\vec{v}_0$, we have $\vec{v}(0) = \vec{v}_0$. Use the expression for $\vec{v}_0$ you obtained in part 1.

$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle C_1, -gt + C_2 \rangle$

Since $\vec{v}(0) = \vec{v}_0 = \langle v_0 \cos\alpha, v_0 \sin\alpha \rangle$: $\langle v_0 \cos\alpha, v_0 \sin\alpha \rangle = \langle C_1, C_2 \rangle$

$\Rightarrow \boxed{\vec{v}(t) = \langle v_0 \cos\alpha, v_0 \sin\alpha - gt \rangle}$

4. Now, using your answer from part 3, write an expression for the position $\vec{r}(t)$ of the projectile.
Hint 1. Recall that $\vec{v}(t) = \vec{r}'(t)$. *Hint 2.* Don't forget the constant vector of integration. *Hint 3.* Since the initial point is (x_0, y_0), we have $\vec{r}(0) = \langle x_0, y_0 \rangle$.

$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle (v_0 \cos\alpha)t + D_1, (v_0 \sin\alpha)t - \frac{1}{2}gt^2 + D_2 \rangle$

Since $\vec{r}(0) = \langle x_0, y_0 \rangle$: $\langle x_0, y_0 \rangle = \langle D_1, D_2 \rangle$

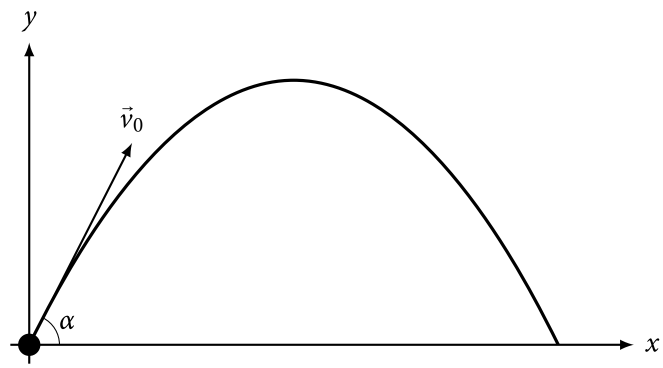
$\Rightarrow \boxed{\vec{r}(t) = \langle x_0 + (v_0 \cos\alpha)t, y_0 + (v_0 \sin\alpha)t - \frac{1}{2}gt^2 \rangle}$

5. Expand the vector equation you obtained in part 4 to write parametric equations (i.e. $x = \ldots, y = \ldots$) for the trajectory of the projectile.

$x = x_0 + (v_0 \cos\alpha)t, \quad y = y_0 + (v_0 \sin\alpha)t - \frac{1}{2}gt^2$

3 Distance traveled by a projectile

- Let us now work under the assumption that the initial point of the projectile is $(0,0)$: in other words, $x_0 = 0$, $y_0 = 0$.



6. The horizontal distance traveled by the projectile d is the value of x when $y = 0$. Why?

The projectile hits the ground when $y=0$.

7. Set $y = 0$ and $y_0 = 0$ to your expression for y in part 5. Solve for t.

$y = y_0 + (v_0 \sin\alpha)t - \frac{1}{2}gt^2$

Setting $y=0, y_0=0$: $0 = (v_0 \sin\alpha)t - \frac{1}{2}gt^2$

$0 = t\left(v_0 \sin\alpha - \frac{1}{2}gt\right)$

$\Rightarrow \boxed{t = 0, \frac{2v_0 \sin\alpha}{g}}$

8. Use your answer in part 7 to obtain an expression for the horizontal distance traveled by the projectile.

The projectile hits the ground at $t = \frac{2v_0 \sin\alpha}{g}$.

$\Rightarrow$ horizontal distance $= 0 + (v_0 \cos\alpha)\left(\frac{2v_0 \sin\alpha}{g}\right) = \frac{2v_0^2 \sin\alpha \cos\alpha}{g} = \frac{v_0^2 \sin 2\alpha}{g}$.

4 Other questions

9. Again, assume that the initial point of the projectile is $(0,0)$. What is the maximum vertical height achieved by the projectile?

10. Take your parametric equation for x in part 5 and solve for t. Plug this back into your parametric equation for y. You should have an expression for y in terms of x. This gives you an idea of how the projectile's trajectory looks like in the xy-plane. What shape does the trajectory take?