SM223 – Calculus III with Optimization
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Lesson 33. Constrained Optimization

0 Review

Example 1. Find the local maxima and minima and saddle points of $f(x,y) = (x^2 + y)e^{y/2}$.

$f_x(x,y) = 2xe^{y/2}$ $\quad f_y(x,y) = (x^2+y)(\frac{1}{2}e^{y/2}) + (1)e^{y/2} = e^{y/2}(\frac{1}{2}x^2 + \frac{1}{2}y + 1)$

critical points: $\left.\begin{array}{l} 2xe^{y/2} = 0 \\ (\frac{1}{2}x^2 + \frac{1}{2}y + 1)e^{y/2} = 0 \end{array}\right\} \Rightarrow (0,-2)$ is a critical point of f

$f_{xx}(x,y) = 2e^{y/2}$ $\quad f_{xy}(x,y) = (2xe^{y/2})(\frac{1}{2}) = xe^{y/2}$ $\quad f_{yy}(x,y) = e^{y/2}(\frac{1}{2}) + (\frac{1}{2}e^{y/2})(\frac{1}{2}x^2 + \frac{1}{2}y + 1)$
$= e^{y/2}(\frac{1}{4}x^2 + \frac{1}{4}y + 1)$

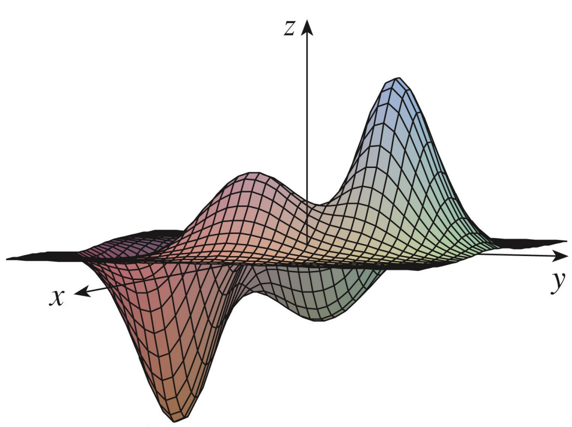
$f_{xx}(0,-2) = 2e^{-1}$ $\quad f_{xy}(0,-2) = 0$ $\quad f_{yy}(0,-2) = e^{-1}(\frac{1}{4}(0) + \frac{1}{4}(-2) + 1) = \frac{1}{2}e^{-1}$

$\Rightarrow D = f_{xx}f_{yy} - f_{xy}^2 = (2e^{-1})(\frac{1}{2}e^{-1}) - 0 = e^{-2} > 0$

$D > 0,\ f_{xx} > 0 \Rightarrow (0,-2)$ is a local minimum.

1 Absolute minima and maxima

- (a,b) is an **absolute minimum** if $f(a,b) \leq f(x,y)$ for all (x,y) in the domain of f
- (a,b) is an **absolute maximum** if $f(a,b) \geq f(x,y)$ for all (x,y) in the domain of f
- Every absolute minimum is a local minimum
- However, a local minimum is not necessarily an absolute ~~max~~ min imum!



- Same statements apply for absolute/local maxima

2 Optimization with 1 equality constraint

Example 2. Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.

distance from (x,y,z) to $(1,0,-2)$: $d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$

model: $\min \sqrt{(x-1)^2+y^2+(z+2)^2}$ s.t. $x+2y+z=4$ $\xrightarrow{\text{substitution}}$ $\min \sqrt{(x-1)^2+y^2+(4-x-2y+2)^2}$

$\iff \min \sqrt{(x-1)^2+y^2+(6-x-2y)^2}$

$\iff \min \underbrace{(x-1)^2+y^2+(6-x-2y)^2}_{f(x,y)}$

$f_x(x,y) = 2(x-1)+2(6-x-2y)(-1)$

$f_y(x,y) = 2y+2(6-x-2y)(-2)$

$\left.\begin{array}{l} f_x=0 \\ f_y=0 \end{array}\right\} \Rightarrow \left(\frac{11}{6},\frac{5}{3}\right)$ is a critical point of f

$f_{xx}(x,y) = 2+2=4$

$f_{xy}(x,y) = 4$

$f_{yy}(x,y) = 2+8=10$

$\Big\} \Rightarrow D = 40-16 = 24 > 0$, $f_{xx} > 0$ $\Rightarrow \left(\frac{11}{6},\frac{5}{3}\right)$ is a local minimum of f

Since $\left(\frac{11}{6},\frac{5}{3}\right)$ is the only local min. of f, it is the absolute min. of f

$\Rightarrow$ shortest distance $= \sqrt{\left(\frac{11}{6}-1\right)^2+\left(\frac{5}{3}\right)^2+\left(6-\frac{11}{6}-2\left(\frac{5}{3}\right)\right)^2} = \frac{5\sqrt{6}}{6}$

Example 3. A rectangular box is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

Volume of box w/ dimensions $x,y,z = xyz$

Surface area of box w/ dimensions $x,y,z = 2xy+2yz+2xz$

model: $\max xyz$ s.t. $2xy+2yz+2xz=12$, $x,y,z>0$ $\xrightarrow{\text{substitution}}$ $\max \overbrace{\frac{xy(6-xy)}{x+y}}^{f(x,y)}$ s.t. $x,y>0$

sides of a box should have positive length

$f_x(x,y) = \frac{y^2(6-2xy-x^2)}{(x+y)^2}$ $\quad f_y(x,y) = \frac{x^2(6-2xy-y^2)}{(x+y)^2}$

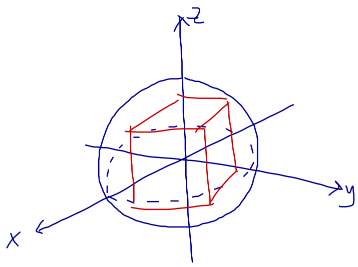
critical points: $f_x=0$, $f_y=0$ $\Rightarrow (0,0), (\sqrt{2},\sqrt{2}), (-\sqrt{2},-\sqrt{2})$ are critical pts. of f.

Since x,y are constrained to be >0, the only critical point to consider is $(\sqrt{2},\sqrt{2})$.

By the physical nature of the problem, there must be an absolute maximum volume, which must occur at a critical point $\Rightarrow (\sqrt{2},\sqrt{2})$ is an absolute maximum of f

$\Rightarrow$ Maximum volume of box $= f(\sqrt{2},\sqrt{2}) = \frac{2(4)}{2\sqrt{2}} = \frac{4}{\sqrt{2}}$.

Example 4. Find the maximum volume of a rectangular box inscribed in a sphere of radius 1.



Sphere centered at origin w/radius 1 has equation

$$x^2+y^2+z^2=1$$

Box centered at origin w/edges parallel to coordinate axes and vertex (x,y,z) in the first orthant has sides of length $2x, 2y, 2z$ $\Rightarrow$ volume $=(2x)(2y)(2z)=8xyz$

model: $\max\ 8xyz$ $\xrightarrow{\text{substitution}}$ $\max\ \overbrace{8xy\sqrt{1-x^2-y^2}}^{f(x,y)}$

s.t. $x^2+y^2+z^2=1$ $\qquad$ s.t. $0<x,y<1$

Since vertex is in first orthant and within the sphere $\longrightarrow$ $0<x,y,z<1$

$$f_x(x,y)=-8xy\left(\tfrac{1}{2}(1-x^2-y^2)^{-1/2}\right)(-2x)+8y\sqrt{1-x^2-y^2}=-8x^2y\left(1-x^2-y^2\right)^{-1/2}+8y\sqrt{1-x^2-y^2}$$

$$f_y(x,y)=-8xy^2\left(1-x^2-y^2\right)^{-1/2}+8x\sqrt{1-x^2-y^2}$$

Setting $\left.\begin{matrix} f_x=0 \\ f_y=0 \end{matrix}\right\} \Rightarrow$ critical pts: $\left(-\frac{\sqrt3}{3},-\frac{\sqrt3}{3}\right),\left(-\frac{\sqrt3}{3},\frac{\sqrt3}{3}\right),\left(\frac{\sqrt3}{3},-\frac{\sqrt3}{3}\right),\left(\frac{\sqrt3}{3},\frac{\sqrt3}{3}\right)$

Since x,y are constrained to be >0, the only critical point we need to consider is $\left(\frac{\sqrt3}{3},\frac{\sqrt3}{3}\right)$. Since there must be an absolute maximum volume and it must occur at a critical point, $\left(\frac{\sqrt3}{3},\frac{\sqrt3}{3}\right)$ must be an absolute maximum of f.

$\Rightarrow$ Max. volume $=f\left(\frac{\sqrt3}{3},\frac{\sqrt3}{3}\right)=8\left(\frac{\sqrt3}{3}\right)\left(\frac{\sqrt3}{3}\right)\sqrt{1-\frac{1}{3}-\frac{1}{3}}=\frac{8}{3\sqrt3}$