Lesson 41. Double Integrals over Rectangles

0 Warm up

Example 1. Find the value of

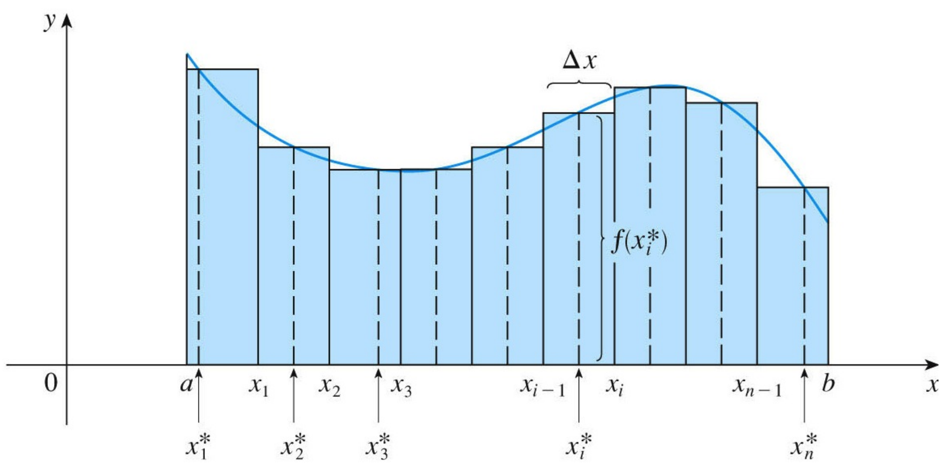
(a) $\displaystyle\sum_{i=2}^{4} i =$

(b) $\displaystyle\sum_{i=1}^{3}\sum_{j=1}^{2} ij =$

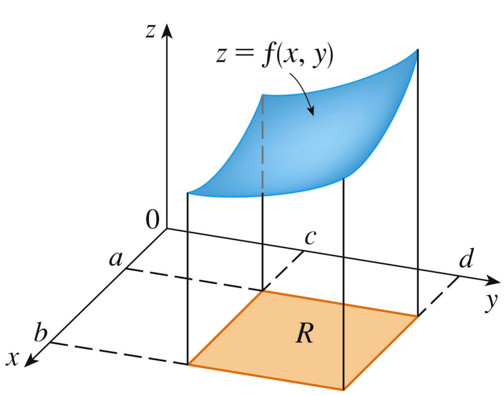
1 Review: area and integrals

- The definite integral of a single-variable function:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$



2 Volume and double integrals

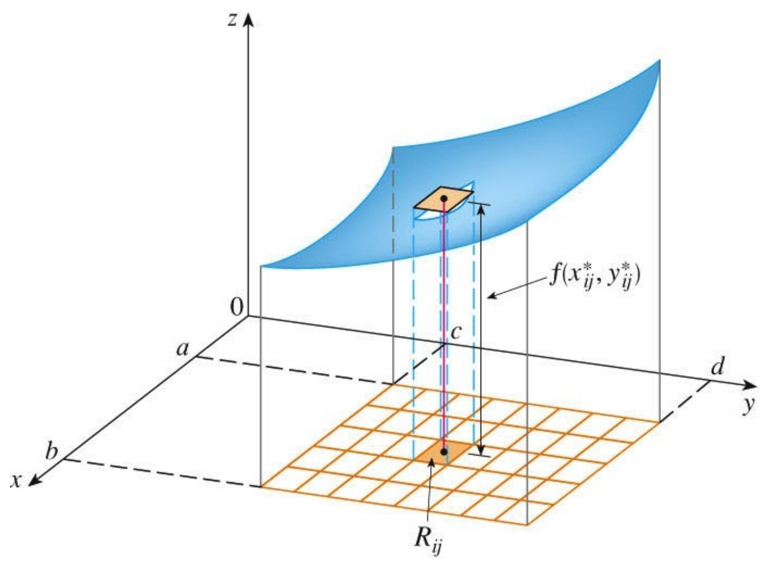


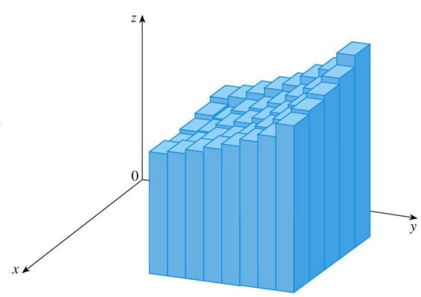
- Let R be a rectangle in the xy-plane:

$$R = [a,b]\times[c,d] = \{(x,y) : a \le x \le b, c \le y \le d\}$$

- Let $f(x,y)$ be a function of two variables
- What is the volume of the solid above R and below the graph of f?

- Idea:
    - Divide R into subrectangles of equal area ΔA
        - Grid with m columns (x-direction) and n rows (y-direction)
    - For each subrectangle R_{ij}:
        - Choose a **sample point** (x_{ij}^*, y_{ij}^*)
        - Compute the volume of the (thin) rectangular box with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$.
    - Add the volumes of all these rectangular boxes





- Estimated volume:

    - This is called a **double Riemann sum**

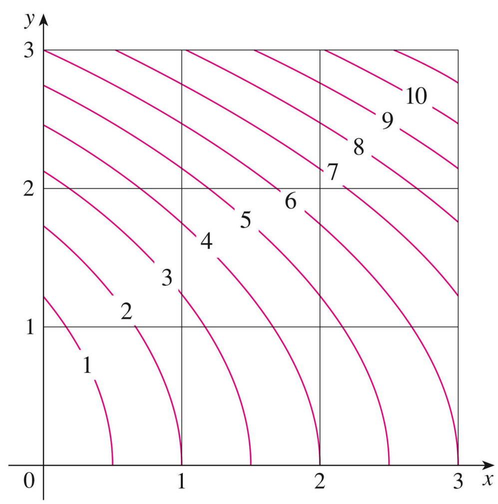
- The **double integral** of f over the rectangle R is

- How do we choose sample points in each subrectangle?
    - Upper right corner
    - Lower left corner
    - **Midpoint rule**: center of subrectangle

- If $f(x, y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is

Example 2. Estimate the volume of the solid that lies above the square $R = [0,2] \times [0,2]$ and below $f(x,y) = 16 - x^2 - 2y^2$. Use a Riemann sum with $m = 2$ and $n = 2$. Use the upper right corners as sample points.

Example 3. Below is a contour map for a function f on the square $R = [0,3] \times [0,3]$. Use a Riemann sum with $m = 3$ and $n = 3$ to estimate the value of $\iint_R f(x,y)\,dA$. Use the midpoint rule to take sample points.



3 Average value

- The **average value** of a function of two variables defined on a rectangle R is

Example 4. Estimate the average value of the function f in Example 3 on R.