Lesson 42. Iterated Integrals

- **Partial integration** with respect to x: $\int_a^b f(x,y)\,dx$
  - Regard y as a constant (i.e., fixed, coefficient, etc.)
  - Integrate $f(x,y)$ with respect to x from $x = a$ to $x = b$
  - Results in an expression in terms of y
- Partial integration with respect to y defined in a similar way
- **Iterated integrals**: work from the inside out
  - $\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$
    - Integrate first with respect to x from $x = a$ to $x = b$ (keeping y constant)
    - Integrate resulting expression in y with respect to y from $y = c$ to $y = d$
  - $\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$
    - Integrate first with respect to $\square$ from $\square$
    - Integrate resulting expression in $\square$ with respect to $\square$ from $\square$

Example 1. Evaluate: (a) $\int_0^3 \int_1^2 x^2 y\,dy\,dx$, (b) $\int_1^2 \int_0^3 x^2 y\,dx\,dy$

- **Fubini's theorem for rectangles.** If $R = [a,b] \times [c,d]$, then:

  - (f needs to satisfy some conditions, e.g. f is continuous on R)
  - Double integrals over rectangles can be evaluated using iterated integrals
  - Order of integration does not matter!

Example 2. Evaluate $\iint_R (x - 3y^2)\, dA$, where $R = [0, 2] \times [1, 2]$.

Example 3. Find the volume of the solid that is bounded by the surface $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

Example 4. Evaluate $\int_0^1 \int_0^1 ye^{xy}\, dy\, dx$.