SM223 – Calculus III with Optimization Fall 2012
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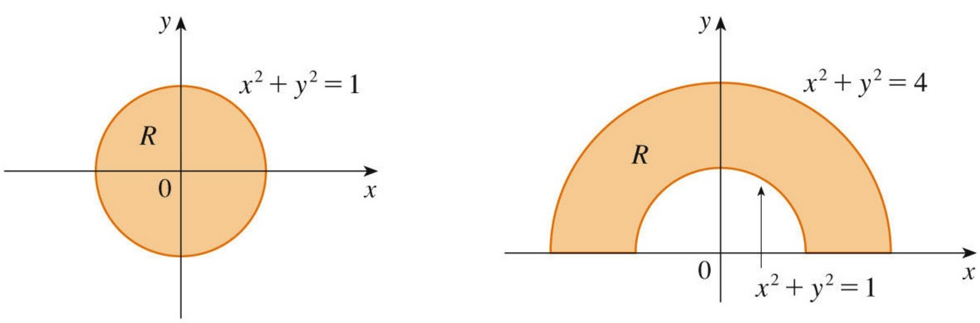
Lesson 46. Double Integrals in Polar Coordinates

0 Warm up

Example 1. Find a polar equation for the curve represented by the Cartesian equation $4y^2 = x$.

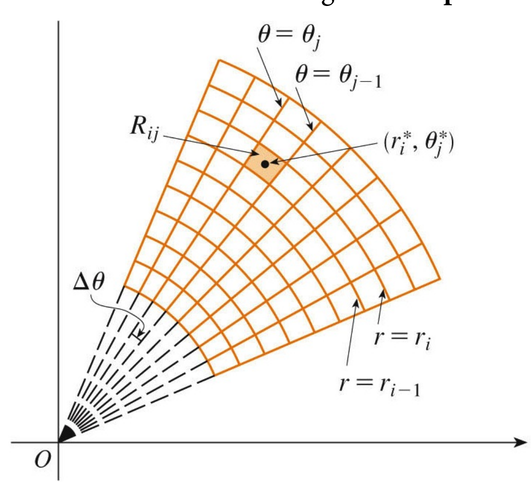
1 Changing to polar coordinates in a double integral

- Idea:
  - Some regions are hard to express in terms of rectangular coordinates, but easily described using polar coordinates



(a) $R = \{(r, \theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$ (b) $R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

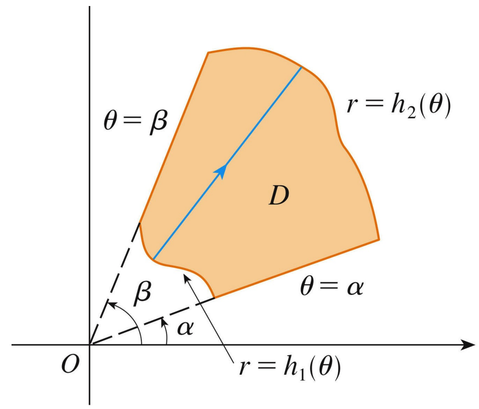
  - How do we integrate in polar coordinates? Divide regions into **polar subrectangles**



- If D is a polar region of the form

$$D = \left\{(r, \theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\right\}$$

then



- Substitute $x = r\cos\theta$ and $y = r\sin\theta$ into $f(x, y)$
- Replace dA with $r\,dr\,d\theta$
- **Don't forget the additional factor r!**

Example 2. Evaluate $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \sin(x^2 + y^2)\,dy\,dx$ by converting to polar coordinates.

Example 3. Evaluate $\iint_D (x^2 + y^2)\,dA$, where D is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Example 4. Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

Example 5. Set up a double integral in polar coordinates that is equal to the volume of the solid that lies underneath the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.