

Lesson 47. Application: Mass and Centers of Mass

1 Review

Example 1. Find the volume of the solid enclosed by the hyperboloid $-x^2 - y^2 + z^2 = 1$ and the plane $z = 2$.

2 Definitions

- Suppose we have a **lamina** or thin plate that occupies a region D of the xy-plane
- $\rho(x, y)$ = **density** of the plate at point (x, y) (units: mass per unit area)
- The **mass** of the lamina is given by
$$m = \iint_D \rho(x, y)\, dA$$
- The **moment of the lamina about the x-axis** is
$$M_x = \iint_D y\rho(x, y)\, dA$$
- The **moment of the lamina about the y-axis** is
$$M_y = \iint_D x\rho(x, y)\, dA$$
- The **center of mass** of the lamina is $(\overline{x}, \overline{y})$, where
$$\overline{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\rho(x, y)\, dA \qquad \overline{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\rho(x, y)\, dA$$
    - The lamina behaves as if the entire mass is concentrated at its center of mass

3 Examples

Example 2. Find the mass and center of mass of a triangular lamina with vertices $(0,0)$, $(1,0)$, and $(0,1)$ if the density function is $\rho(x,y) = x + y$. Just set up the integrals, do not evaluate.

Example 3. Find the mass and center of mass of a lamina that is bounded by the parabolas $y = x^2$ and $x = y^2$ if the density function is $\rho(x,y) = \sqrt{x}$. Just set up the integrals, do not evaluate.

Example 4. The boundary of a lamina consists of the semicircle $y = \sqrt{4 - x^2}$ together with the x-axis. Find the center of mass of the lamina if the density at any point is proportional to its distance from the origin, that is, $\rho(x,y) = k\sqrt{x^2 + y^2}$. Use polar coordinates. Just set up the integrals, do not evaluate.