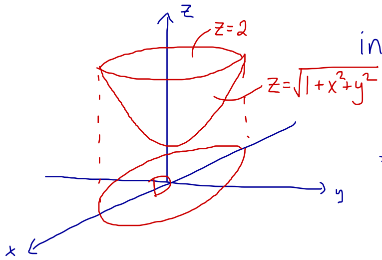
Lesson 47. Application: Mass and Centers of Mass

1 Review

Example 1. Find the volume of the solid enclosed by the hyperboloid $-x^2 - y^2 + z^2 = 1$ and the plane $z = 2$.



intersection of 2 surfaces (top): $-x^2-y^2+2^2=1$

$\iff x^2+y^2=3$

$\Rightarrow$ region of integration: filled circle centered at origin w/ radius $\sqrt{3}$

$\frac{d}{dr}\left[(1+r^2)^{3/2}\right] = \frac{3}{2}(1+r^2)^{1/2}(2r) = 3r\sqrt{1+r^2}$

$$V = \iint_D \left(2-\sqrt{1+x^2+y^2}\right) dA = \int_0^{2\pi}\int_0^{\sqrt{3}} \left(2r - r\sqrt{1+r^2}\right) dr\, d\theta = \int_0^{2\pi} \left[r^2 - \frac{1}{3}(1+r^2)^{3/2}\right]_{r=0}^{r=\sqrt{3}} d\theta$$

$$= \int_0^{2\pi} \left[3 - \frac{1}{3}(4)^{3/2} - 0 + \frac{1}{3}\right] d\theta = \int_0^{2\pi} \left[3 - \frac{8}{3} + \frac{1}{3}\right] d\theta = \int_0^{2\pi} \frac{2}{3}\, d\theta = \frac{4\pi}{3}$$

2 Definitions

- Suppose we have a **lamina** or thin plate that occupies a region D of the xy-plane
- $\rho(x, y)$ = **density** of the plate at point (x, y) (units: mass per unit area)
- The **mass** of the lamina is given by

$$m = \iint_D \rho(x, y)\, dA$$

- The **moment of the lamina about the x-axis** is

$$M_x = \iint_D y\rho(x, y)\, dA$$

- The **moment of the lamina about the y-axis** is

$$M_y = \iint_D x\rho(x, y)\, dA$$

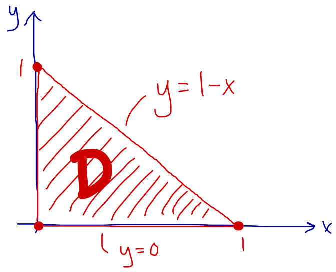
- The **center of mass** of the lamina is $(\overline{x}, \overline{y})$, where

$$\overline{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\rho(x, y)\, dA \qquad \overline{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\rho(x, y)\, dA$$

  - The lamina behaves as if the entire mass is concentrated at its center of mass

3 Examples

Example 2. Find the mass and center of mass of a triangular lamina with vertices $(0,0)$, $(1,0)$, and $(0,1)$ if the density function is $\rho(x,y) = x + y$. Just set up the integrals, do not evaluate.

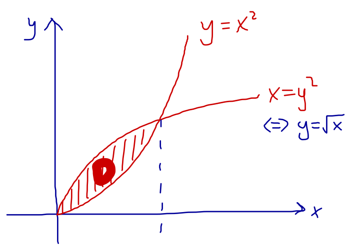


$$m = \iint_D \rho(x,y)\,dA = \int_0^1 \int_0^{1-x} (x+y)\,dy\,dx$$

$$\bar{x} = \frac{1}{m}\iint_D x\,\rho(x,y)\,dA = \frac{1}{m}\int_0^1 \int_0^{1-x} x(x+y)\,dy\,dx$$

$$\bar{y} = \frac{1}{m}\iint_D y\,\rho(x,y)\,dA = \frac{1}{m}\int_0^1 \int_0^{1-x} y(x+y)\,dy\,dx$$

Example 3. Find the mass and center of mass of a lamina that is bounded by the parabolas $y = x^2$ and $x = y^2$ if the density function is $\rho(x,y) = \sqrt{x}$. Just set up the integrals, do not evaluate.

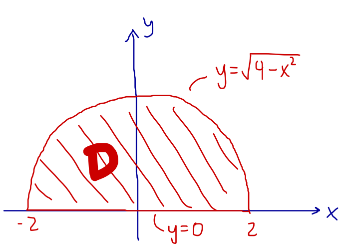


$$m = \int_0^1 \int_{x^2}^{\sqrt{x}} \sqrt{x}\,dy\,dx$$

$$\bar{x} = \frac{1}{m}\int_0^1 \int_{x^2}^{\sqrt{x}} x\sqrt{x}\,dy\,dx$$

$$\bar{y} = \frac{1}{m}\int_0^1 \int_{x^2}^{\sqrt{x}} y\sqrt{x}\,dy\,dx$$

Example 4. The boundary of a lamina consists of the semicircle $y = \sqrt{4 - x^2}$ together with the x-axis. Find the center of mass of the lamina if the density at any point is proportional to its distance from the origin, that is, $\rho(x,y) = k\sqrt{x^2 + y^2}$. Use polar coordinates. Just set up the integrals, do not evaluate.



$$m = \iint_D \rho(x,y)\,dA = \int_0^{\pi}\int_0^2 k\sqrt{r^2}\,r\,dr\,d\theta = \int_0^{\pi}\int_0^2 kr^2\,dr\,d\theta$$

$$\bar{x} = \frac{1}{m}\iint_D x\rho(x,y)\,dA = \int_0^{\pi}\int_0^2 (r\cos\theta)(kr)\,r\,dr\,d\theta = \int_0^{\pi}\int_0^2 kr^3\cos\theta\,dr\,d\theta$$

$$\bar{y} = \frac{1}{m}\iint_D y\rho(x,y)\,dA = \int_0^{\pi}\int_0^2 (r\sin\theta)(kr)\,r\,dr\,d\theta = \int_0^{\pi}\int_0^2 kr^3\sin\theta\,dr\,d\theta.$$