SM223 – Calculus III with Optimization	Fall 2012
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Lesson 51. Mass and Center of Mass, Revisited

1 Definitions

- Suppose a solid object occupies a region E in space
- Let $\rho(x, y, z)$ = density of the object at (x, y, z)
- The **mass** of the object is

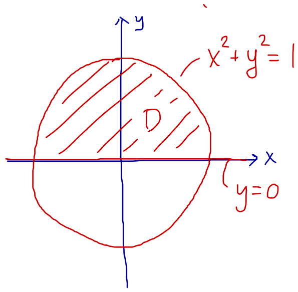
$$\boxed{m = \iiint_E \rho(x,y,z)\, dV}$$

- The **center of mass** of the object is located at $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{1}{m}\iiint_E x\rho(x,y,z)\, dV \qquad \bar{y} = \frac{1}{m}\iiint_E y\rho(x,y,z)\, dV \qquad \bar{z} = \frac{1}{m}\iiint_E z\rho(x,y,z)\, dV$$

2 Problems

Example 1. Set up integrals to find the mass and center of mass of the solid E, where E is the solid above the xy-plane and bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = y$ and $z = 0$, and the density function is $\rho(x, y, z) = 1 + x + y + z$.



$$m = \iiint_E \rho(x,y,z)\, dV = \int_{-1}^{1}\int_0^{\sqrt{1-x^2}}\int_0^y (1+x+y+z)\, dz\, dy\, dx$$

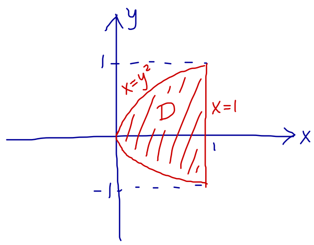
$$\bar{x} = \frac{1}{m}\iiint_E x\rho(x,y,z)\, dV = \frac{1}{m}\int_{-1}^{1}\int_0^{\sqrt{1-x^2}}\int_0^y x(1+x+y+z)\, dz\, dy\, dx$$

$$\bar{y} = \frac{1}{m}\iiint_E y\rho(x,y,z)\, dV = \frac{1}{m}\int_{-1}^{1}\int_0^{\sqrt{1-x^2}}\int_0^y y(1+x+y+z)\, dz\, dy\, dx$$

$$\bar{z} = \frac{1}{m}\iiint_E z\rho(x,y,z)\, dV = \frac{1}{m}\int_{-1}^{1}\int_0^{\sqrt{1-x^2}}\int_0^y z(1+x+y+z)\, dz\, dy\, dx$$

Example 2. Let E be the solid bounded by the parabolic cylinder $x = y^2$ and the planes $z = x$, $z = 0$, and $x = 1$, and let the density function be $\rho(x, y, z) = y^2$.

a. Set up integrals for the mass and center of mass of E.
b. Use your calculator to evaluate the integrals you set up in part a.



$$m = \iiint_E \rho(x,y,z)\,dV = \iint_D \int_0^x y^2\,dz\,dA$$

$$= \int_{-1}^{1}\int_{y^2}^{1}\int_0^x y^2\,dz\,dx\,dy = \frac{4}{21}$$

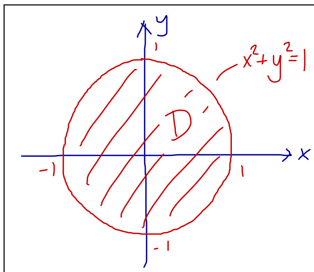
$$\bar{x} = \frac{1}{m}\iiint_E x\rho(x,y,z)\,dV = \frac{1}{m}\int_{-1}^{1}\int_{y^2}^{1}\int_0^x xy\;dz\,dx\,dy = \left(\frac{21}{4}\right)\left(\frac{4}{27}\right) = \frac{7}{9}$$

$$\bar{y} = \frac{1}{m}\iiint_E y\rho(x,y,z)\,dV = \frac{1}{m}\int_{-1}^{1}\int_{y^2}^{1}\int_0^x y^2\,dz\,dx\,dy = \left(\frac{21}{4}\right)(0) = 0$$

$$\bar{z} = \frac{1}{m}\iiint_E z\rho(x,y,z)\,dV = \frac{1}{m}\int_{-1}^{1}\int_{y^2}^{1}\int_0^x yz\;dz\,dx\,dy = \left(\frac{21}{4}\right)\left(\frac{2}{27}\right) = \frac{7}{18}$$

Example 3. Let E be the solid bounded by the sphere $x^2 + y^2 + z^2 = 1$ and the xy-plane, and let the density function be $\rho(x, y, z) = (x - 1)^2$.

a. Set up integrals for the mass and center of mass of E.
b. Use your calculator to evaluate the integrals you set up in part a.



$$m = \iiint_E \rho(x,y,z)\,dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} (x-1)^2\,dz\,dy\,dx = \frac{4\pi}{5}$$

$$\bar{x} = \frac{1}{m}\iiint_E x\rho(x,y,z)\,dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} x(x-1)^2\,dz\,dy\,dx = \left(\frac{5}{4\pi}\right)\left(-\frac{4\pi}{15}\right) = -\frac{1}{3}$$

$$\bar{y} = \frac{1}{m}\iiint_E y\rho(x,y,z)\,dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} y(x-1)^2\,dz\,dy\,dx = \left(\frac{5}{4\pi}\right)(0) = 0$$

$$\bar{z} = \frac{1}{m}\iiint_E z\rho(x,y,z)\,dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} z(x-1)^2\,dz\,dy\,dx = \left(\frac{5}{4\pi}\right)\left(\frac{7\pi}{24}\right) = \frac{35}{96}$$