Lesson 4. The Cross Product

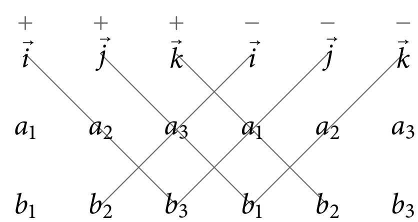
1 In this lesson...

- Computing the cross product
- The right-hand rule
- Areas and the cross product
- Volumes and the scalar triple product

2 Computing the cross product

- If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of $\vec{a}$ and $\vec{b}$ is

- Note: $\vec{a} \times \vec{b}$ is a vector (unlike the dot product)
- The cross product is only defined for 3D vectors
- Mnemonic for taking the cross product:

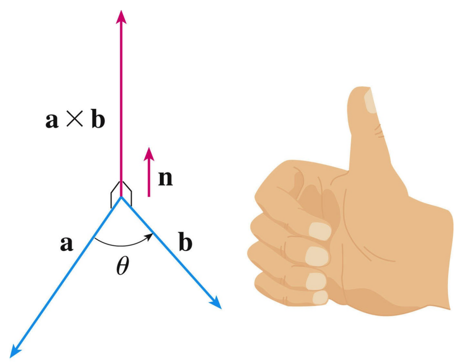


Example 1. Let $\vec{a} = \langle 1, 3, 4 \rangle$ and $\vec{b} = \langle 2, 7, -5 \rangle$. Find $\vec{a} \times \vec{b}$.

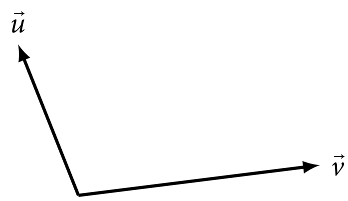
3 The right-hand rule

- The vector $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.
- Orthogonal which way? **Right-hand rule**
  - Curl fingers of right hand from $\vec{a}$ to $\vec{b}$

  $\Rightarrow$ Thumb points in direction of $\vec{a} \times \vec{b}$



Example 2. Find the direction of $\vec{u} \times \vec{v}$.



Example 3. Find two unit vectors orthogonal to both $\vec{a} = 2\vec{j} - \vec{k}$ and $\vec{b} = \vec{i} + 4\vec{j}$.

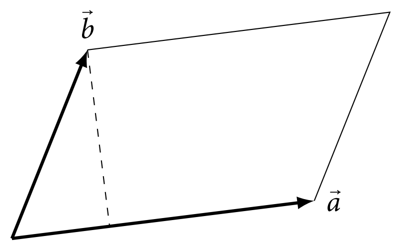
4 Areas and the cross product

- What about the magnitude of $\vec{a} \times \vec{b}$?
- If θ is the angle between $\vec{a}$ and $\vec{b}$, then

- $\sin\theta = 0$ when $\theta =$

$\Rightarrow$ Two nonzero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if

- $|\vec{a} \times \vec{b}|$ = the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$:

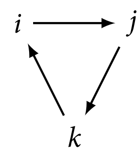


Example 4. Find the area of the triangle with vertices $P(1, 4, 2)$, $Q(-2, 5, -1)$, and $R(1, 3, 1)$.

- Cross products between $\vec{i}$, $\vec{j}$ and $\vec{k}$ are pretty easy to remember:

$$\vec{i} \times \vec{j} = \vec{k} \qquad \vec{j} \times \vec{k} = \vec{i} \qquad \vec{k} \times \vec{i} = \vec{j}$$
$$\vec{j} \times \vec{i} = -\vec{k} \qquad \vec{k} \times \vec{j} = -\vec{i} \qquad \vec{i} \times \vec{k} = -\vec{j}$$

  - Mnemonic:



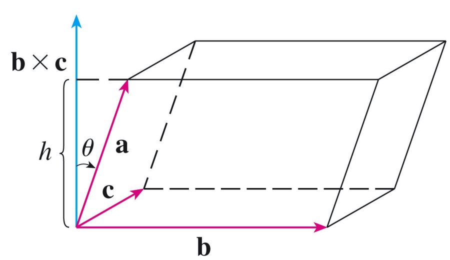
- **Properties of cross products:** if $\vec{a}, \vec{b}, \vec{c}$ are vectors and c is a scalar:

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \qquad (\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$$
$$(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b}) \qquad \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$$
$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \qquad \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

- The cross product is not commutative, i.e., $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$
- The cross product is not associative either, i.e. $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

5 Volumes and the scalar triple product

- The **scalar triple product** of $\vec{a}$, $\vec{b}$, and $\vec{c}$ is
- $|\vec{a} \cdot (\vec{b} \times \vec{c})|$ = the volume of the **parallelpiped** determined by $\vec{a}$, $\vec{b}$, and $\vec{c}$:



Example 5. Find the volume of the parallelpiped determined by $\vec{a} = \langle 1, 2, 3 \rangle$, $\vec{b} = \langle -1, 1, 2 \rangle$, and $\vec{c} = \langle 2, 1, 4 \rangle$.

- If we find that the volume of the parallelpiped determined by $\vec{a}$, $\vec{b}$, and $\vec{c}$ is 0, then
  - In other words, the vectors are