Solutions.

Problem 1.

$$\frac{\partial v}{\partial s} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial s} = (2x\sin y + y^2 e^{xy})(1) + (x^2\cos y + xye^{xy} + e^{xy})(t)$$
$$\frac{\partial v}{\partial t} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial t} = (2x\sin y + y^2 e^{xy})(2) + (x^2\cos y + xye^{xy} + e^{xy})(s)$$

Note that when $s = 0$ and $t = 1$, we have that $x = 2$ and $y = 0$. Therefore,

$$\left.\frac{\partial v}{\partial s}\right|_{\substack{s=0\\t=1}} = (0+0)(1) + (4+0+1)(1) = 5$$
$$\left.\frac{\partial v}{\partial t}\right|_{\substack{s=0\\t=1}} = (0+0)(2) + (4+0+1)(0) = 0$$

Problem 2.

$$\begin{aligned}\frac{dP}{dt} &= \frac{\partial P}{\partial L}\frac{dL}{dt} + \frac{\partial P}{\partial K}\frac{dK}{dt}\\ &= [1.47(0.65)L^{-0.35}K^{0.35}]\frac{dL}{dt} + [1.47(0.35)L^{0.65}K^{-0.65}]\frac{dK}{dt}\\ &= [1.47(0.65)(30)^{-0.35}(8)^{0.35}](-2) + [1.47(0.35)(30)^{0.65}(8)^{-0.65}](0.5) \qquad \text{(note the units!)}\\ &= -0.5958\end{aligned}$$

Problem 3.

a. $\nabla f(x,y) = \langle 2xy, x^2 + \frac{1}{2}y^{-\frac{1}{2}}\rangle$

b. The direction in question is $\vec{v} = \langle 3,2\rangle$. The unit vector in the same direction as $\vec{v}$ is $\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{13}}\langle 3,2\rangle$. So, the directional derivative of f at $(2,1)$ in the direction $\vec{v}$ is

$$D_{\vec{u}}f(2,1) = \nabla f(2,1)\cdot\vec{u} = \langle 4, \frac{9}{2}\rangle \cdot \frac{1}{\sqrt{13}}\langle 3,2\rangle = \frac{21}{\sqrt{13}}$$

Therefore, at $(2,1)$ in the direction towards $(5,3)$, the slope of f is $\frac{21}{\sqrt{13}}$.

c. The maximum rate of change of f at $(2,1)$ is $|\nabla f(2,1)| = \frac{\sqrt{145}}{2}$.

d. The direction in which the maximum rate of change of f at $(2,1)$ occurs is $\nabla f(2,1) = \langle 4, \frac{9}{2}\rangle$.

Problem 4.

Let $F(x,y,z) = xy + yz + xz = 5$. Then, the surface in question is given by the equation $F(x,y,z) = 5$, and the normal vector to the surface at $(1,2,1)$ is $\nabla F(1,2,1)$:

$$\nabla F(x,y,z) = \langle y+z, x+z, x+y\rangle \qquad \Rightarrow \qquad \nabla F(1,2,1) = \langle 3,2,3\rangle$$

Therefore, an equation of the tangent plane to the surface at $(1, 2, 1)$ is

$$3(x-1) + 2(y-2) + 3(z-1) = 0$$

and parametric equations of the normal line to the surface at $(1, 2, 1)$ are

$$x = 1 + 3t \qquad y = 2 + 2t \qquad z = 1 + 3t$$

Problem 5.

First, find all the first partial derivatives:

$$f_x(x, y) = 4y - 4x^3 \qquad f_y(x, y) = 4x - 4y^3$$

Next, find the critical points by solving the following system of equations:

$$\begin{aligned} 4y - 4x^3 &= 0 \qquad &\Rightarrow \qquad y = x^3 \qquad (1) \\ 4x - 4y^3 &= 0 \qquad &\Rightarrow \qquad x = y^3 \qquad (2) \end{aligned}$$

Substituting (1) into (2), we get $x = x^9$, which implies that $x = -1, 0, 1$. Plugging this back into (1), we get the following critical points $(-1, -1)$, $(0, 0)$, and $(1, 1)$.

Now, find the second partial derivatives:

$$f_{xx}(x, y) = -12x^2 \qquad f_{yy}(x, y) = -12y^2 \qquad f_{xy}(x, y) = 4$$

Now we can perform the second derivative test:

$$D(x, y) = (-12x^2)(-12y^2) - 4^2 = 144x^2y^2 - 16$$

- $(-1, -1)$: $D(-1, -1) = 128 > 0$, $f_{xx}(-1, -1) = -12 < 0$. Therefore, $(-1, -1)$ is a local maximum.
- $(0, 0)$: $D(0, 0) = -16 < 0$. Therefore, $(0, 0)$ is a saddle point.
- $(1, 1)$: $D(1, 1) = 128 > 0$, $f_{xx}(1, 1) = -12 < 0$. Therefore, $(1, 1)$ is a local maximum.

Problem 6.

The optimization model for this problem is

$$\begin{aligned} \text{maximize} \quad & xyz \\ \text{subject to} \quad & x + 2y + 2z = 108 \quad (x, y, z > 0) \end{aligned}$$

In the notation we used in class for the Lagrange multiplier method, $f(x, y, z) = xyz$, $g(x, y, z) = x + 2y + 2z$ and $k = 108$. The gradients are

$$\nabla f(x, y, z) = \langle yz, xz, xy \rangle \qquad \nabla g(x, y, z) = \langle 1, 2, 2 \rangle$$

So, the Lagrange multiplier equations are

$$\begin{aligned} yz &= \lambda \qquad (1) \\ xz &= 2\lambda \qquad (2) \\ xy &= 2\lambda \qquad (3) \\ x + 2y + 2z &= 108 \qquad (4) \end{aligned}$$

(1) and (2) imply $xz = 2yz$, which implies $x = 2y$, since z must be strictly positive. (2) and (3) imply $xz = xy$, which implies $z = y$, since x also must be strictly positive. Substituting into (4), we obtain $2y + 2y + 2y = 108$, or $y = 18$. Tracing our steps backwards, we obtain $x = 36$ and $z = 18$. Therefore, we have one candidate for a maximum/minimum to our optimization model, $(36, 18, 18)$, whose value is $f(36, 18, 18) = 11664$.

We can determine if $f(36, 18, 18) = 11664$ is a minimum or maximum by testing another point that satsifies the constraint $x + 2y + 2z = 108$, such as $(104, 1, 1)$. Note that $f(104, 1, 1) = 104$, so it must be the case that $f(36, 18, 18) = 11664$ is a maximum.