# Lesson 3. The Dot Product

## 1 In this lesson...

- Definition and properties of the dot product
- Dot products and angles between vectors
- Direction angles and direction cosines
- Projections
- Practice with vectors and dot products

## 2 The dot product

- We know how to multiply a vector by a scalar
- Can we multiply two vectors together? Yes!
- If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, the **dot product** of $\vec{a}$ and $\vec{b}$ is

- Note that $\vec{a} \cdot \vec{b}$ is a scalar
- The dot product of vectors in $\mathbb{R}^2$ is defined similarly: if $\vec{a} = \langle a_1, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2 \rangle$, then

**Example 1.**

a. $\langle -1, 7 \rangle \cdot \langle 6, 2 \rangle =$

b. $\langle 2, 4, 1 \rangle \cdot \langle -1, 3, 1 \rangle =$

c. $(-\vec{i} + 3\vec{k} + 4\vec{j}) \cdot (\vec{i} - 3\vec{k}) =$

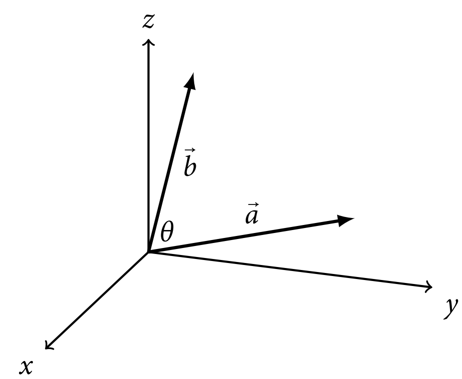
- **Properties of the dot product**

$$\vec{a} \cdot \vec{a} = |a|^2 \qquad (c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b})$$
$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \qquad \vec{0} \cdot \vec{a} = 0$$
$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

- The dot product behaves very similarly to ordinary products of real numbers

## 3 Dot products and angles

- The **angle** $\theta$ between two vectors $\vec{a}$ and $\vec{b}$:




  - We always take the angle so that $0 \leq \theta \leq \pi$

- If $\vec{a}$ and $\vec{b}$ are scalar multiples of one another, we say that the vectors are **parallel**

  - If $\vec{a}$ and $\vec{b}$ are parallel, then $\theta =$

- If $\theta$ is the angle between vectors $\vec{a}$ and $\vec{b}$, then

$\Rightarrow$ If $\theta$ is the angle between nonzero vectors $\vec{a}$ and $\vec{b}$, then

**Example 2.** Find the angle between vectors $\vec{a} = \langle 2, -1, 3 \rangle$ and $\vec{b} = \langle -3, 2, 5 \rangle$.

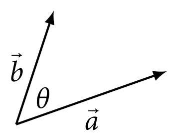
- Two nonzero vectors $\vec{a}$ and $\vec{b}$ are called **perpendicular** or **orthogonal** if the angle between them is $\theta = \pi/2$
- Suppose $\vec{a}$ and $\vec{b}$ are nonzero

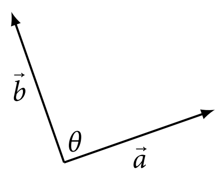
  - If $\vec{a}$ and $\vec{b}$ are perpendicular, then $\vec{a} \cdot \vec{b} =$
  - If $\vec{a} \cdot \vec{b} = 0$, then $\cos\theta =$ and so $\theta =$

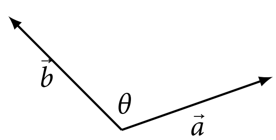
$\Rightarrow$ Two vectors $\vec{a}$ and $\vec{b}$ are orthogonal if and only if

**Example 3.** Show that $2\vec{i} - \vec{j} + 2\vec{k}$ is perpendicular to $5\vec{i} + 2\vec{j} - 4\vec{k}$.

- The dot product measures the extent to which $\vec{a}$ and $\vec{b}$ point in the same direction

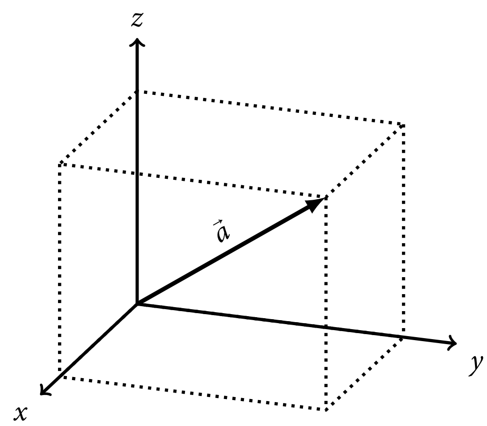
    - 


    - 


    - 

## 4 Direction angles and direction cosines

- **Direction angles** for vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$:

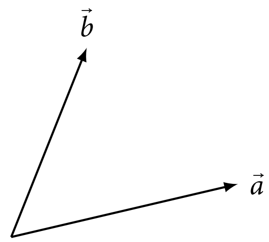


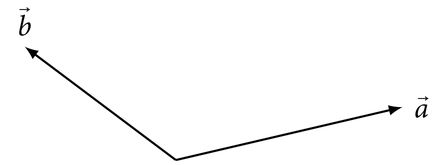

- Again, we take $\alpha, \beta\, \gamma$ always to be in $[0, \pi]$
- Remember that if $\theta$ is the angle between $\vec{a}$ and $\vec{b}$, then $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$
- **Direction cosines**
  - $\cos\alpha =$
  - $\cos\beta =$
  - $\cos\gamma =$
- Note that we can write $\frac{1}{|\vec{a}|}\vec{a} =$

**Example 4.** Find the direction angles of $\vec{a} = \langle 2, 1, 3 \rangle$.

## 5 Projections

- **Vector projection** of $\vec{b}$ onto $\vec{a}$:




  - Denoted by $\text{proj}_{\vec{a}}\vec{b}$
  - "Shadow" of $\vec{b}$ onto $\vec{a}$

- **Scalar projection** of $\vec{b}$ onto $\vec{a}$ = signed magnitude of $\text{proj}_{\vec{a}}\vec{b}$
  - Also called the **component** of $\vec{b}$ along $\vec{a}$
  - Denoted by $\text{comp}_{\vec{a}}\vec{b}$

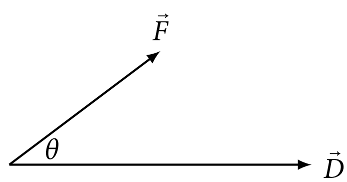
- The scalar and vector projections can be computed using dot products:
  - $\text{comp}_{\vec{a}}\vec{b} =$
  - $\text{proj}_{\vec{a}}\vec{b} =$

**Example 5.** Find the scalar projection and vector projection of $\vec{b} = \langle 3, 4 \rangle$ onto $\vec{a} = \langle 2, 1 \rangle$.

**Example 6.** Draw $\vec{a}$, $\vec{b}$, and $\text{proj}_{\vec{a}}\vec{b}$ from Example 5. Is the drawing what you expected?

- The **work** done by a constant force $\vec{F}$ in moving an object along a displacement vector $\vec{D}$ is defined as

$$W = (\text{component of } \vec{F} \text{ along } \vec{D})(\text{distance moved})$$




$\Rightarrow W =$

**Example 7.** A force $\vec{F} = 5\vec{i} - 2\vec{j} + 3\vec{k}$ moves a particle from the point $P(2, 0, -1)$ to the point $Q(6, 2, 4)$. Find the work done.

## 6 Practice!

**Example 8.** Find the scalar projection and vector projection of $\vec{b} = \langle 1, 1, 2 \rangle$ onto $\vec{a} = \langle -2, 3, 1 \rangle$.

**Example 9.** Find a unit vector that is orthogonal to both $\langle 2, 0, -1 \rangle$ and $\langle 0, 1, -1 \rangle$.

**Example 10.** Determine whether the given vectors are orthogonal, parallel, or neither:

a. $\vec{a} = \langle 4, 5, -2 \rangle$, $\vec{b} = \langle 3, -1, 5 \rangle$
b. $\vec{u} = 9\vec{i} - 6\vec{j} + 3\vec{k}$, $\vec{v} = -6\vec{i} + 4\vec{j} - 2\vec{k}$