

## Lesson 11. Multivariable Functions

### 1 This lesson...

- How do functions of many variables work? What do they look like graphically?

### 2 Functions of 2 variables

- A **function  $f$  of 2 variables**
  - takes an ordered pair  $(x, y)$  of real numbers as input
  - outputs a unique real number  $f(x, y)$
- The **domain  $D$**  of  $f$  is the set of allowable inputs to  $f$ 
  - If  $f$  is given by a formula and its domain is not explicitly specified, then the domain of  $f$  is the set of all  $(x, y)$  for which the formula is well-defined
- The **range** of  $f$  is the set of values that  $f$  takes on

**Example 1.** Let  $f(x, y) = \frac{\sqrt{x + y + 1}}{x - 1}$ .

- a. What is  $f(3, 2)$ ?
- b. What is the domain of  $f$ ?

**Example 2.** In 1928, using economic data published by the government, Charles Cobb and Paul Douglas modeled production output  $P(L, K)$  as a function of the amount of labor involved  $L$  and the amount of capital invested  $K$ :

$$P(L, K) = 1.01L^{0.75}K^{0.25}$$

This function (in a more general form) is known as the *Cobb-Douglas production function*.

Find  $P(120, 20)$ . In words, what does  $P(120, 20)$  mean?

- Functions are not always represented by explicit formulas, as the next example shows

**Example 3.** The *wind-chill index*  $W(T, v)$  is a subjective temperature that is a function of the actual temperature  $T$  (in  $^{\circ}\text{C}$ ) and wind speed  $v$  (in km/h), as given by the table below:

		Wind speed (km/h)											
		$v$	5	10	15	20	25	30	40	50	60	70	80
Actual temperature ( $^{\circ}\text{C}$ )	$T$	5	4	3	2	1	1	0	-1	-1	-2	-2	-3
	0	-2	-3	-4	-5	-6	-6	-7	-8	-9	-9	-10	
	-5	-7	-9	-11	-12	-12	-13	-14	-15	-16	-16	-17	
	-10	-13	-15	-17	-18	-19	-20	-21	-22	-23	-23	-24	
	-15	-19	-21	-23	-24	-25	-26	-27	-29	-30	-30	-31	
	-20	-24	-27	-29	-30	-32	-33	-34	-35	-36	-37	-38	
	-25	-30	-33	-35	-37	-38	-39	-41	-42	-43	-44	-45	
	-30	-36	-39	-41	-43	-44	-46	-48	-49	-50	-51	-52	
	-35	-41	-45	-48	-49	-51	-52	-54	-56	-57	-58	-60	
	-40	-47	-51	-54	-56	-57	-59	-61	-63	-64	-65	-67	

- Find  $W(-15, 40)$ . In words, what does  $W(-15, 40)$  mean?
- Define the function  $h(T) = W(T, 40)$ . Describe the behavior of  $h$ .

### 3 Functions of $n$ variables

- A function  $f$  of  $n$  variables

- takes an ordered tuple  $(x_1, \dots, x_n)$  of real numbers as input
- outputs a unique real number  $f(x_1, \dots, x_n)$

**Example 4.** Anteater-Bugs produces  $n$  types of beers. It costs  $c_i$  to produce one bottle of type  $i$  beer ( $i = 1, \dots, n$ ). Let  $C(x_1, x_2, \dots, x_n)$  be the total cost of producing  $x_1$  bottles of type 1 beer,  $x_2$  bottles of type 2 beer,  $\dots$ , and  $x_n$  bottles of type  $n$  beer.

Write a formula for  $C(x_1, x_2, \dots, x_n)$ .

### 4 Graphs of functions in 2 variables

- Let  $f$  be a function of 2 variables with domain  $D$
- The **graph** of  $f$  is the set of all points  $(x, y, z)$  in  $\mathbb{R}^3$  such that  $z = f(x, y)$  and  $(x, y)$  is in  $D$

**Example 5.** Sketch the graph of  $f(x, y) = x^2 + 2y^2 + 1$  by first drawing its traces for  $z = -1, 0, 1, 2, 3, 4$ .



