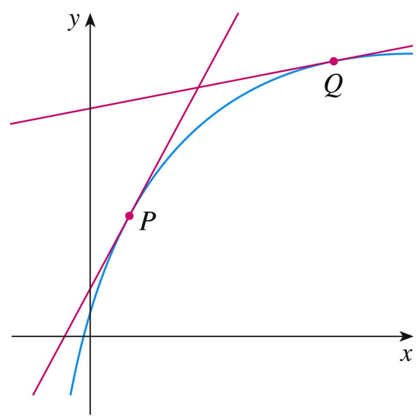
Lesson 13. Partial Derivatives

1 This lesson...

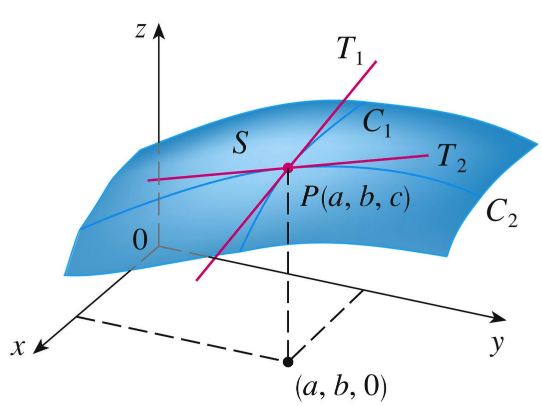
- Definition of partial derivative
- Computing partial derivatives
- Higher derivatives
- Practice!

2 Definition

- Derivatives of single-variable functions
    - Instantaneous rate of change
    - Slope of tangent line



- How can we get similar things for multivariable functions? Partial derivatives
- Idea: let $f(x, y)$ be a function of 2 variables
    - Fix the value of y to $b \Rightarrow g(x) = f(x, b)$ is a function in 1 variable x
    - Take the derivative of $g(x) = f(x, b)$ with respect to x
    - **This gives us the rate of change of $f(x, y)$ with respect to x when $y = b$**
    - Repeat, but with fixing the value of x and taking the derivative with respect to y



- The **partial derivative of $f(x, y)$ with respect to x** is

- The **partial derivative of $f(x, y)$ with respect to y** is

- In words, $\partial f / \partial x$ is

- In words, $\partial f / \partial y$ is

Example 1. Here is the wind-chill index function $W(T, v)$ from Lesson 11:

Wind speed (km/h)

Actual temperature (°C)

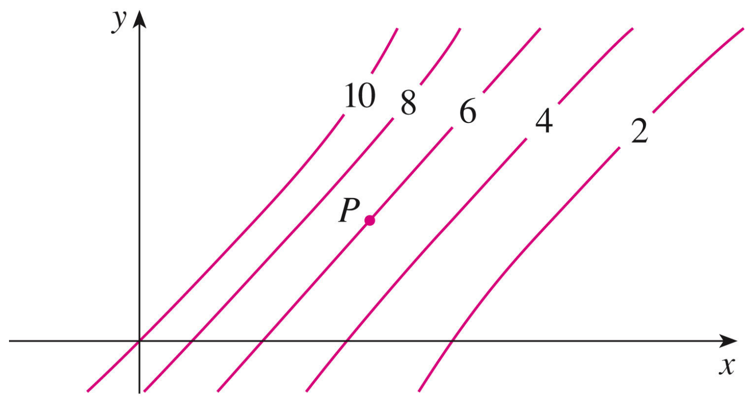
T \ v	5	10	15	20	25	30	40	50	60	70	80
5	4	3	2	1	1	0	−1	−1	−2	−2	−3
0	−2	−3	−4	−5	−6	−6	−7	−8	−9	−9	−10
−5	−7	−9	−11	−12	−12	−13	−14	−15	−16	−16	−17
−10	−13	−15	−17	−18	−19	−20	−21	−22	−23	−23	−24
−15	−19	−21	−23	−24	−25	−26	−27	−29	−30	−30	−31
−20	−24	−27	−29	−30	−32	−33	−34	−35	−36	−37	−38
−25	−30	−33	−35	−37	−38	−39	−41	−42	−43	−44	−45
−30	−36	−39	−41	−43	−44	−46	−48	−49	−50	−51	−52
−35	−41	−45	−48	−49	−51	−52	−54	−56	−57	−58	−60
−40	−47	−51	−54	−56	−57	−59	−61	−63	−64	−65	−67

a. Estimate $W_T(-15, 40)$.
b. Give a practical interpretation of this value.

Example 2. Here are the level curves for a function $f(x, y)$. Determine whether the following partial derivatives are positive or negative at the point P.

a. f_x

b. f_y



3 Computing partial derivatives

- Let $f(x, y)$ be a function of 2 variables
- To find f_x, treat y as a constant and differentiate $f(x, y)$ with respect to x
- To find f_y, treat x as a constant and differentiate $f(x, y)$ with respect to y

Example 3. Let $f(x, y) = 3x^3 + 2x^2y^3 - 5y^2$. Find $f_x(2, 1)$ and $f_y(2, 1)$.

Example 4. Let $f(x, y) = \frac{x}{y}$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Example 5. Let $f(x, y) = \sin\left(\frac{x}{1+y}\right)$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

4 Higher derivatives

- We can take partial derivatives of partial derivatives
- The **second partial derivatives of** $f(x, y)$ are
  - $f_{xx} =$
  - $f_{xy} =$
  - $f_{yx} =$
  - $f_{yy} =$
- **Clairaut's theorem.** Suppose f is defined on a disk D that contains the point (a, b).

  If f_{xy} and f_{yx} are continuous on D, then
- We can take third partial derivatives (e.g. f_{xxy}), fourth partial derivatives (e.g. f_{yxyy}), etc.

5 Examples

Do as many as you can!

Problem 1. Use the table of values of $f(x, y)$ to estimate the values of $f_x(3, 2)$ and $f_y(3, 2)$.

x \ y	1.8	2.0	2.2
2.5	12.5	10.2	9.3
3.0	18.1	17.5	15.9
3.5	20.0	22.4	26.1

Problem 2. Consider the level curves given in Example 2. Determine whether the following partial derivatives are positive or negative at the point P.

a. f_{xx}

b. f_{yy}

c. f_{xy}

Problem 3. Let $f(x, y) = \arctan(y/x)$. Find $f_x(2, 3)$.

Problem 4. Let $f(x, y, z) = \dfrac{y}{x + y + z}$. Find $f_y(2, 1, -1)$.

(Partial derivatives of functions of 3 or more variables are found the same way: regard all but one variable as constant, and take the derivative with respect to the remaining variable.)

Problem 5. Let $f(x, y, z) = \sqrt{\sin^2 x + \sin^2 y + \sin^2 z}$. Find $f_x(0, 0, \pi/4)$.

Problem 6. Find all the second partial derivatives of $f(x, y) = x^4 y - 2x^3 y^2$.

Problem 7. Let $f(x,y) = \cos(x^2 y)$. Verify that Clairaut's theorem holds: $f_{xy} = f_{yx}$.

Problem 8. Let $f(x,y) = \sin(2x + 5y)$. Find f_{yxy}.

Problem 9. Find all the second partial derivatives of $f(x,y) = \ln(ax + by)$.

Problem 10. The temperature at a point (x, y) on a flat metal plate is given by $T(x, y) = 60/(1 + x^2 + y^2)$, where T is measured in °C and x, y in meters. Find the rate of change in temperature with respect to distance at the point $(2, 1)$ in the x-direction and the y-direction.

Problem 11. The average energy E (in kcal) neeeded for a lizard to walk or run a distance of 1 km has been modeled by the equation

$$E(m, v) = 2.65m^{0.66} + \frac{3.5m^{0.75}}{v}$$

where m is the body mass of the lizard (in grams) and v is its speed (in km/h). Calculate $E_m(400, 8)$ and $E_v(400, 8)$ and interpret your answers.

Problem 12. Cobb and Douglas used the equation $P(L, K) = 1.01L^{0.75}K^{0.25}$ to model the productivity of the American economy from 1899 to 1922, where L is the amount of labor and K is the amount of capital.

a. Calculate P_L and P_K.
b. Find the rate of change in productivity with respect to labor and capital in the year 1899, when $L = 100$ and $K = 100$. Interpret the results.
c. Do the same for the year 1920, when $L = 194$ and $K = 407$.
d. In the year 1920, which would have benefited production more, an increase in capital investment or an increase in spending on labor?

Problem 13. Consider the contour map of a function f given below. Are the following derivatives at the given point positive or negative?

a. f_x
b. f_y
c. f_{xx}
d. f_{yy}
e. f_{xy}