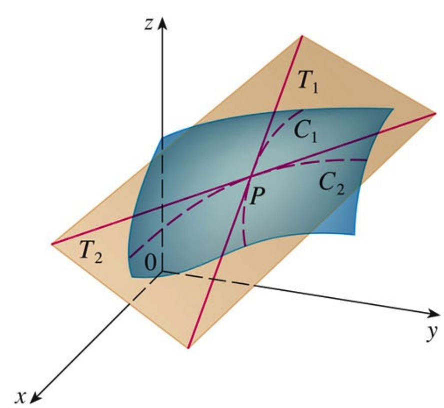
Lesson 14. Tangent Planes and Linear Approximations

0 Warm up

Example 1. Find an equation of the plane that passes through $(-1, 1, 5)$ and is perpendicular to the vector $\langle 2, 4, -3 \rangle$.

1 Tangent planes



- Let S be a surface with equation $z = f(x, y)$
- Let $P(x_0, y_0, z_0)$ be a point on S
  - So $z_0 =$
- Let T_1 and T_2 be the tangent lines at P in the x- and y-directions, respectively
- The **tangent plane** to the surface S at point P is the plane that contains both tangent lines T_1 and T_2
- The tangent plane must have an equation of the form $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ or equivalently,
- If this is the equation of the tangent plane, its intersection with the plane $y = y_0$ must be the tangent line T_1
- Setting $y = y_0$, we obtain
- Looking at this equation A must be
- Similarly, B must be

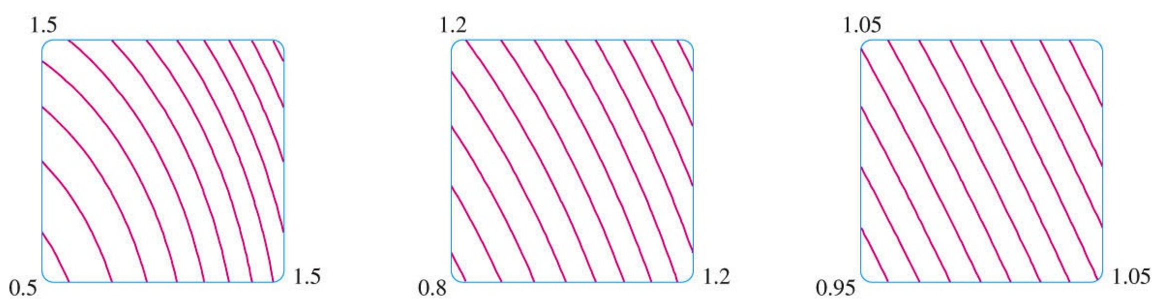
⇒ An equation of the tangent plane to the surface $z = f(x, y)$ at point $P(x_0, y_0, z_0)$ is

Example 2. Find the tangent plane to the surface $z = x^2 + xy + 3y^2$ at the point $(1, 1, 5)$.

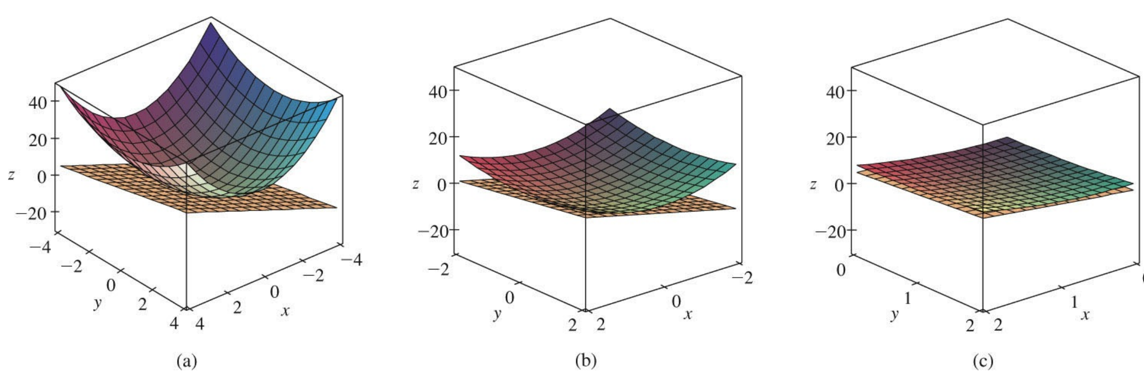
2 Linear approximations

- What do the level curves of a plane look like?

- As we zoom in on the level curves of an arbitrary surface, they start to look more and more like equally spaced parallel lines
  - For example: $f(x, y) = 2x^2 + y^2$



⇒ We can use tangent planes to approximate function values



- The **linear approximation** of f at (a, b) is

  - Compare to equation for tangent plane above: use $x_0 = a$, $y_0 = b$, $z_0 = f(a, b)$

Example 3. Find the linear approximation of $f(x, y) = xe^{xy}$ at $(1, 0)$. Use it to approximate $f(1.1, -0.2)$.

Example 4. Here is the wind-chill index function $W(T, v)$ we have seen in previous lessons:

Wind speed (km/h)

Actual temperature (°C)

T \ v	5	10	15	20	25	30	40	50	60	70	80
5	4	3	2	1	1	0	−1	−1	−2	−2	−3
0	−2	−3	−4	−5	−6	−6	−7	−8	−9	−9	−10
−5	−7	−9	−11	−12	−12	−13	−14	−15	−16	−16	−17
−10	−13	−15	−17	−18	−19	−20	−21	−22	−23	−23	−24
−15	−19	−21	−23	−24	−25	−26	−27	−29	−30	−30	−31
−20	−24	−27	−29	−30	−32	−33	−34	−35	−36	−37	−38
−25	−30	−33	−35	−37	−38	−39	−41	−42	−43	−44	−45
−30	−36	−39	−41	−43	−44	−46	−48	−49	−50	−51	−52
−35	−41	−45	−48	−49	−51	−52	−54	−56	−57	−58	−60
−40	−47	−51	−54	−56	−57	−59	−61	−63	−64	−65	−67

Once upon a time, we estimated $W_T(-15, 40) \approx 1.3$. In a similar fashion, we can estimate $W_v(-15, 40) \approx -0.15$. Find the linear approximation of $W(T, v)$ at $(-15, 40)$. Use it to approximate $W(-12, 45)$.

- Why bother with linear approximations?
  - Desert island
  - More importantly: **linear functions** (functions of the form $f(x, y) = ax + by$) are much easier to deal with that other types of functions
    - $\Rightarrow$ Linear approximations form the basis of many algorithms for complex problems
- Disclaimer: equations for tangent planes and linear approximations above do not necessarily apply when the partial derivatives of f are not continuous

3 Differentials

- Suppose we want to find the difference between $f(x, y)$ and $f(a, b)$: $f(x, y) - f(a, b)$
- Recall that the linear approximation of f at (a, b) is

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- Letting $dx = x - a$ and $dy = y - b$ and rearranging terms, we get

- The **differential** of f is

- So we can approximate the difference between $f(x, y)$ and $f(a, b)$ by computing the differential df

Example 5. Let $f(x, y) = x^2 + 3xy - y^2$.

a. Find the differential df.
b. Use the differential to estimate the change in f when x changes from 2 to 2.05 and y changes from 3 to 2.96.
c. Compare your approximation with the actual change.