Name: Notes

SM275 · Mathematical Methods for Economics Fall 2019 · Uhan

Exam 3 – 4 December 2019

Instructions

- You have until the end of the class period to complete this exam.
- You may use your calculator.
- You may consult the SM275 Formula Table given to you with this exam.
- You may not consult any other outside materials (e.g. notes, textbooks, homework, computer).
- **Show all your work.** To receive full credit, your solution must be completely correct, sufficiently justified, and easy to follow.
- Keep this booklet intact.

Problem	Weight	Score
1	2	
2	2	
3	2	
4	2	
5	2	
6	2	
7	2	
8	2	
9	1	
10	1	
11	1	
12	1	
Total		/ 200

For Problems 1 and 2, consider the function

$$f(x_1, x_2, x_3) = 2x_1^2 + x_2^4 - 4x_2 + x_3^2$$

Problem 1. Find the gradient of f.

Problem 2. Find the critical points of f. How many critical points are there? Do not determine whether there is a local minimum, local maximum, or saddle point at each critical point.

For Problems 3 and 4, consider the function $f(x_1, x_2, x_3) = x_1^2 + x_1x_3 + x_2^3 - 12x_2 + 3x_3^2$. The gradient of f is

$$\nabla f(x_1, x_2, x_3) = \begin{bmatrix} 2x_1 + x_3 \\ 3x_2^2 - 12 \\ x_1 + 6x_3 \end{bmatrix}$$

Problem 3. Find the Hessian of f.

Problem 4. One of the critical points of f is $(0, 2, 0)$. Determine whether this critical point is a local minimum, local maximum, or saddle point.

- Take a look at Lesson 16 for examples on how to apply the second derivative test for unconstrained optimization problems.

For Problems 5-7, consider the optimization problem

$$\begin{aligned} \text{minimize/maximize} \quad & -x_1^3 + 3x_1 - x_3^2 \\ \text{subject to} \quad & 3x_1 + 2x_2 = 9 \end{aligned}$$

Problem 5. Write the Lagrangian function for this optimization problem.

Problem 6. The gradient of the Lagrangian function is

$$\nabla L(\lambda, x_1, x_2, x_3) = \begin{bmatrix} -3x_1 - 2x_2 + 9 \\ -3x_1^2 + 3 - 3\lambda \\ -2\lambda \\ -2x_3 \end{bmatrix}$$

Briefly explain why $(0, 1, 3, 0)$ is a constrained critical point.

- Nearly all of you correctly identified that $(0, 1, 3, 0)$ is a solution to a particular system of equations (which system?), but many of you did not explain <u>why</u> $(0, 1, 3, 0)$ is a solution to that system.
- In general, how do you prove that a solution solves a system of equations, without going through the trouble of solving the system? As a hint, consider the following example. Here's a system of equations:

  $$\begin{aligned} 2x + y - 5 &= 0 \\ -x + 2y &= 0 \end{aligned}$$

  A solution to the above system is $(x, y) = (2, 1)$. You can prove this without solving the system by plugging in $(2, 1)$ into the system and making sure all of the equalities hold.

Problem 7. The Hessian of the Lagrangian function is

$$H_L(\lambda, x_1, x_2, x_3) = \begin{bmatrix} 0 & -3 & -2 & 0 \\ -3 & -6x_1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

One of the constrained critical points is $(0, 1, 3, 0)$. Determine whether this constrained critical point leads to a constrained local minimum, constrained local maximum, or constrained saddle point.

- Take a look at Lessons 19 and 20 for examples of how to apply the second derivative test for constrained optimization problems.
- Some hints:
    - For this problem, what is the number of variables n? What is the number of equality constraints k?
    - Recall that the second derivative test for constrained optimization requires the principal minors d_{2k+1} to d_{k+n}. What are $2k+1$ and $k+n$ in this case? (A further hint: you should find that you need 2 principal minors.)
    - Don't forget to multiply each principal minor by $(-1)^k$.

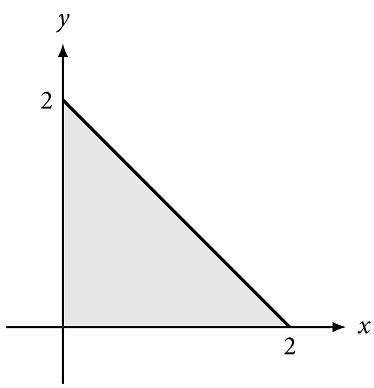
For Problems 8 and 9, consider the following setting.

Professor I. M. Wright is trying to solve the following optimization problem:

$$\begin{aligned} \text{maximize} \quad & f(x,y) = -4x + 2y - x^2 - y^2 \\ \text{subject to} \quad & x + y \leq 2 \\ & x \geq 0, y \geq 0 \end{aligned}$$

Looking at the professor's notes, you see:

- Feasible region:



- f has a single critical point at $(-2, 1)$.
- On the line segment between $(0,0)$ and $(2,0)$, the maximum value of f is 0, achieved at $(0,0)$.
- On the line segment between $(0,2)$ and $(2,0)$, the maximum value of f is 0, achieved at $(0,2)$.

Problem 8. Find the maximum value of f on the line segment between $(0,0)$ and $(0,2)$. Where is the maximum value on this line segment achieved?

- Many of you ended up with the correct maximum value and location of the maximum value, but didn't have sufficient justification.
- In this problem, you are trying to find the maximum value of a single-variable function between two points.
- One straightforward way to do this is to graph the function between the two points, and visually select the maximum value and its location, like we did in class.
- Take a look at Lesson 21 for similar examples.

Problem 9. Using the professor's notes, and your solution to Problem 8, finish solving the optimization problem. Where is the maximum value of f achieved? What is the maximum value of f?

- Recall from Lesson 21: in order to find an extreme value of f over a closed and bounded feasible region:
  1. Find the values at the unconstrained critical points of f.
     (Keep them only if they're in the feasible region.)
  2. Find the extreme values of f on the boundary of the feasible region.
  3. The largest of the values from Steps 1 and 2 is the absolute maximum value.
     The smallest of these values is the absolute minimum value.
- Step 1 was done for you by Professor Wright.
- Step 2 was partially done for you by Professor Wright. Problem 8 asked you to finish Step 2.
- Step 3 is this problem. Collect all the candidate maximum values and locations from Steps 1 and 2, and pick the best one.

For Problems 10-12, consider the following setting.

The Dantzig Company produces doors and windows. The company earns \$10 of profit for each door sold, and \$15 for each window. The market is large enough so that the company can sell all it produces at these prices. Each door requires 1 hour of carpentry and 3 hours of finishing. Each window requires 2 hours of carpentry and 2 hours of finishing. The company has 16 hours of carpentry and 24 hours of finishing available. The company wants to maximize the profit it earns from selling doors and windows.

Problem 10. Formulate the company's problem as a linear program, using the following variables:

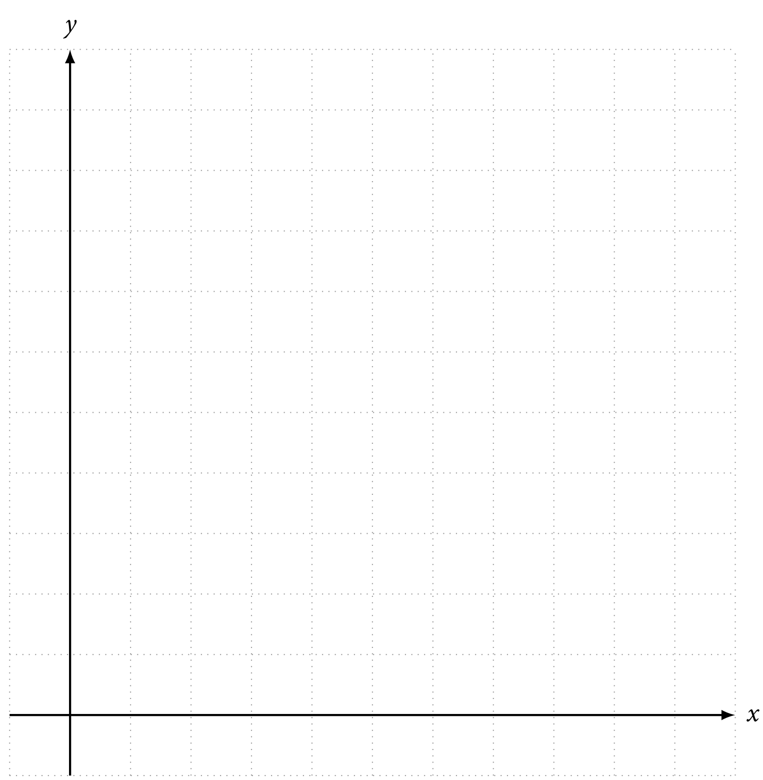
x = number of doors the company should produce
y = number of windows the company should produce

- Your answer should look like this:

  minimize (or maximize) objective function
  subject to constraints

- Many of you forgot to specify whether you are minimizing or maximizing the objective function. Many of you also forgot the nonnegativity constraints. Both of these details are important! The nature of the problem can be vastly different if you don't accurately specify these.

Problem 11. Draw the feasible region of the linear program you wrote in Problem 10.



Problem 12. Solve the linear program you wrote in Problem 10. How many doors and windows should the company produce? What is the maximum profit the company can earn from selling doors and windows?

- You should have found that $(0, 0)$ is one of the corner points of the feasible region you drew in Problem 11. Don't forget to include the value of $(0, 0)$ when comparing corner points in order to find the optimal solution.