Name:

Exam 3 – 4 December 2019

Instructions

- You have until the end of the class period to complete this exam.
- You may use your calculator.
- You may consult the SM275 Formula Table given to you with this exam.
- You may not consult any other outside materials (e.g. notes, textbooks, homework, computer).
- **Show all your work.** To receive full credit, your solution must be completely correct, sufficiently justified, and easy to follow.
- Keep this booklet intact.

Problem	Weight	Score
1	2	
2	2	
3	2	
4	2	
5	2	
6	2	
7	2	
8	2	
9	1	
10	1	
11	1	
12	1	
Total		/ 200

For Problems 1 and 2, consider the function

$$f(x_1, x_2, x_3) = 2x_1^2 + x_2^4 - 4x_2 + x_3^2$$

Problem 1. Find the gradient of f.

Problem 2. Find the critical points of f. How many critical points are there? Do not determine whether there is a local minimum, local maximum, or saddle point at each critical point.

For Problems 3 and 4, consider the function $f(x_1, x_2, x_3) = x_1^2 + x_1x_3 + x_2^3 - 12x_2 + 3x_3^2$. The gradient of f is

$$\nabla f(x_1, x_2, x_3) = \begin{bmatrix} 2x_1 + x_3 \\ 3x_2^2 - 12 \\ x_1 + 6x_3 \end{bmatrix}$$

Problem 3. Find the Hessian of f.

Problem 4. One of the critical points of f is $(0, 2, 0)$. Determine whether this critical point is a local minimum, local maximum, or saddle point.

For Problems 5-7, consider the optimization problem

$$\begin{aligned} \text{minimize/maximize} \quad & -x_1^3 + 3x_1 - x_3^2 \\ \text{subject to} \quad & 3x_1 + 2x_2 = 9 \end{aligned}$$

Problem 5. Write the Lagrangian function for this optimization problem.

Problem 6. The gradient of the Lagrangian function is

$$\nabla L(\lambda, x_1, x_2, x_3) = \begin{bmatrix} -3x_1 - 2x_2 + 9 \\ -3x_1^2 + 3 - 3\lambda \\ -2\lambda \\ -2x_3 \end{bmatrix}$$

Briefly explain why $(0, 1, 3, 0)$ is a constrained critical point.

Problem 7. The Hessian of the Lagrangian function is

$$H_L(\lambda, x_1, x_2, x_3) = \begin{bmatrix} 0 & -3 & -2 & 0 \\ -3 & -6x_1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

One of the constrained critical points is $(0, 1, 3, 0)$. Determine whether this constrained critical point leads to a constrained local minimum, constrained local maximum, or constrained saddle point.

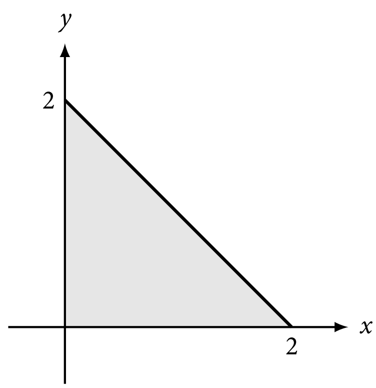
For Problems 8 and 9, consider the following setting.

Professor I. M. Wright is trying to solve the following optimization problem:

$$\begin{aligned} \text{maximize} \quad & f(x,y) = -4x + 2y - x^2 - y^2 \\ \text{subject to} \quad & x + y \leq 2 \\ & x \geq 0, y \geq 0 \end{aligned}$$

Looking at the professor's notes, you see:

- Feasible region:



- f has a single critical point at $(-2, 1)$.
- On the line segment between $(0,0)$ and $(2,0)$, the maximum value of f is 0, achieved at $(0,0)$.
- On the line segment between $(0,2)$ and $(2,0)$, the maximum value of f is 0, achieved at $(0,2)$.

Problem 8. Find the maximum value of f on the line segment between $(0,0)$ and $(0,2)$. Where is the maximum value on this line segment achieved?

Problem 9. Using the professor's notes, and your solution to Problem 8, finish solving the optimization problem. Where is the maximum value of f achieved? What is the maximum value of f?

For Problems 10-12, consider the following setting.

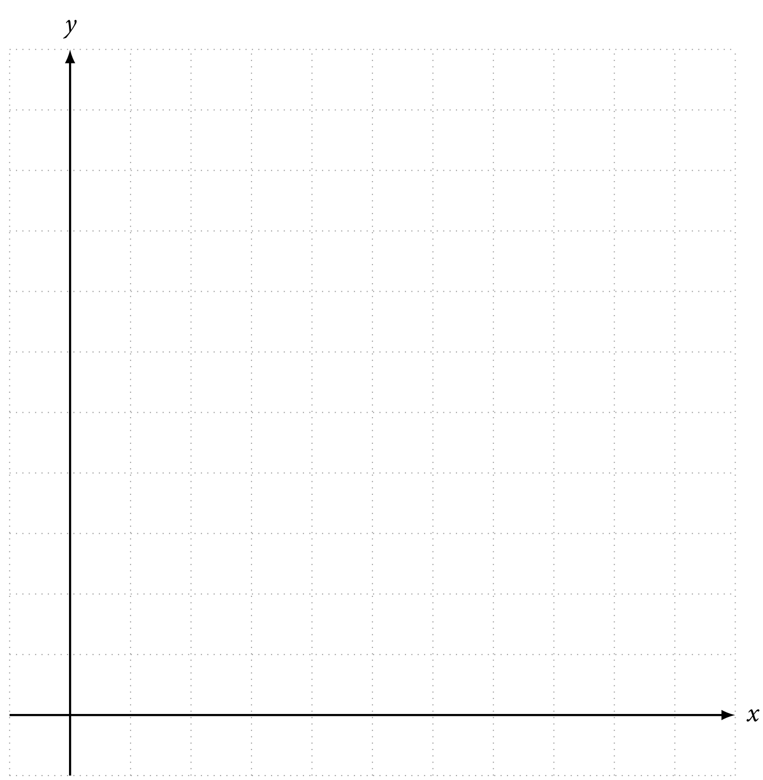
The Dantzig Company produces doors and windows. The company earns \$10 of profit for each door sold, and \$15 for each window. The market is large enough so that the company can sell all it produces at these prices. Each door requires 1 hour of carpentry and 3 hours of finishing. Each window requires 2 hours of carpentry and 2 hours of finishing. The company has 16 hours of carpentry and 24 hours of finishing available. The company wants to maximize the profit it earns from selling doors and windows.

Problem 10. Formulate the company's problem as a linear program, using the following variables:

x = number of doors the company should produce

y = number of windows the company should produce

Problem 11. Draw the feasible region of the linear program you wrote in Problem 10.



Problem 12. Solve the linear program you wrote in Problem 10. How many doors and windows should the company produce? What is the maximum profit the company can earn from selling doors and windows?