Lesson 3. Cobwebs

0 Warm up

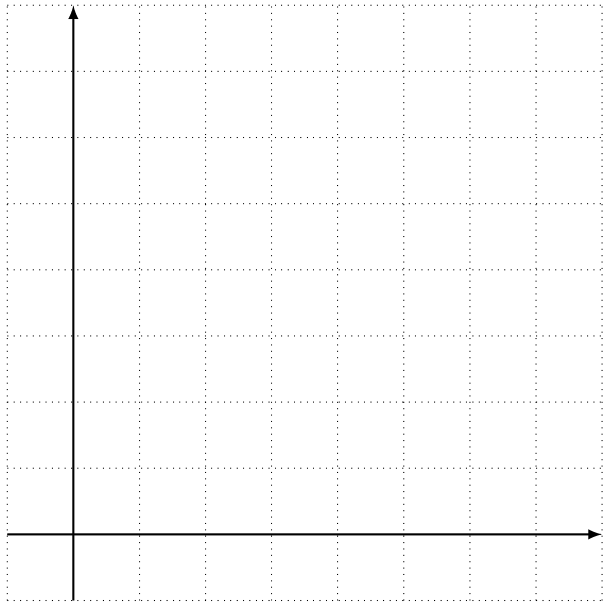
Example 1. Consider the DS $A_{n+1} = \frac{1}{2}A_n + 1$, $n = 0, 1, 2, \ldots$.

a. Let $A_0 = 4$. Compute $A_1, \ldots, A_4$.
b. Let $A_0 = 0$. Compute $A_1, \ldots, A_4$.
c. What are the fixed points of this DS?

1 How to draw a cobweb

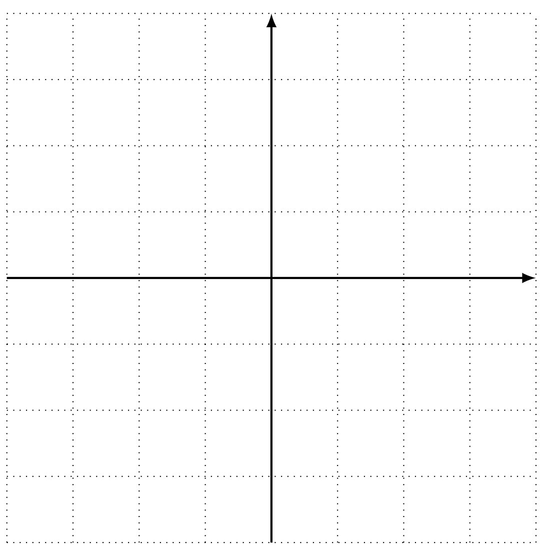
- Sometimes we want to study how a DS behaves with different ICs
- **Cobwebs** are a graphical method for understanding this behavior
- Consider the DS $A_{n+1} = f(A_n)$, $n = 0, 1, 2, \ldots$
- How to draw a cobweb:
    1. Draw the line $y = x$.
    2. Draw the graph of $y = f(x)$.
    3. Pick an initial point A_0 on the x-axis.
    4. Connect $(A_0, 0)$ to (A_0, A_1) with a vertical line.
       Note that $A_1 = f(A_0)$, so (A_0, A_1) is on the graph of $y = f(x)$.
    5. Connect (A_0, A_1) to (A_1, A_1) with a horizontal line.
       Note that (A_1, A_1) is on the graph of $y = x$.
    6. Connect (A_1, A_1) to (A_1, A_2) with a vertical line.
       Note that $A_2 = f(A_1)$, so (A_1, A_2) is on the graph of $y = f(x)$.
    7. Continue in this way.

Example 2. Consider the same DS from Example 1: $A_{n+1} = \frac{1}{2}A_n + 1$, $n = 0, 1, 2, \ldots$.
Draw the cobwebs with $A_0 = 4$ and $A_0 = 0$.



- In the above example, it looks like if A_0 is close to the fixed point $c = 2$, then we eventually end up at the fixed point
- The fixed point $c = 2$ is "attracting" the sequence of points
- We'll come back to this later

Example 3. Consider the same DS from Example 1: $A_{n+1} = 2A_n - 1$, $n = 0, 1, 2, \ldots$.
Draw the cobwebs with $A_0 = 2$ and $A_0 = 0$.

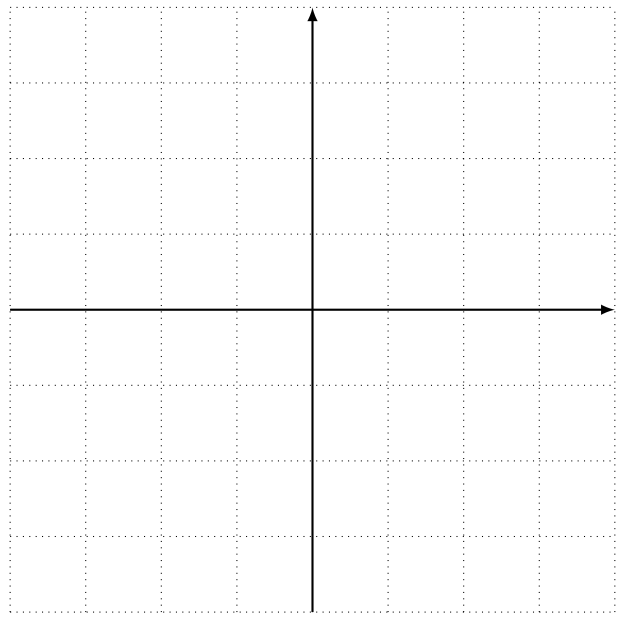


- This time, in the above example, it looks the fixed point $c = 1$ is "repelling" sequence the points

2 Attracting and repelling fixed points

- A fixed point c is **attracting** if whenever A_0 is sufficiently close to c, then $A_n \to c$ as $n \to \infty$
- A fixed point c is **repelling** if no matter how close A_0 is to c, then A_n is eventually far away from c infinitely many times
- A DS may have a fixed point that is neither attracting nor repelling

Example 4. Consider the DS $A_{n+1} = 3A_n - 2$. Find the fixed points. Use cobwebs to determine whether each fixed point is attracting, repelling, or neither.



Example 5. Consider the DS $A_{n+1} = -A_n + 1$. Find the fixed points. Use cobwebs to determine whether each fixed point is attracting, repelling, or neither.