Lesson 8. Fixed Points and Stability for Second Order DS

1 Fixed Points

- A **fixed point** of a second order DS is a number k such that

$$A_n = k \qquad n = 0, 1, 2, \ldots$$

  is a solution to the DS

- Consider the second order linear DS

$$A_{n+2} = aA_{n+1} + bA_n + c \qquad n = 0, 1, 2, \ldots \tag{$\star$}$$

- If k is a fixed point, then

- If $a + b \neq 1$, then we have

- If $a + b = 1$, then we have

2 Stability

- A second order linear DS is **stable** if $\lim_{n\to\infty} A_n$ exists for all initial conditions
- A second order linear DS is **unstable** if $\lim_{n\to\infty} |A_n| = \infty$ for some initial conditions
- There exist DS that are neither stable nor unstable
- General strategy:
  1. Calculate the fixed point
  2. Find the general solution
  3. Examine the behavior of the general solution as $n \to \infty$ to determine if
     - the system is stable/unstable and
     - the fixed point is attracting/repelling

2.1 The case $a + b \neq 1$

- As we saw above, there is a unique fixed point in this case:
- If $A_n \to \frac{c}{1-a-b}$ for all possible initial conditions, the system is stable and the fixed point is **attracting**
- If $|A_n| \to \infty$ for some initial conditions, the system is unstable and the fixed point is **repelling**
- It is possible that neither of these two behaviors is present

Example 1. Consider the DS $A_{n+2} = \frac{5}{6}A_{n+1} - \frac{1}{6}A_n + 1$, $n = 0, 1, 2, \ldots$. The general solution is

$$A_n = c_1\left(\frac{1}{2}\right)^n + c_2\left(\frac{1}{3}\right)^n + 3$$

a. Find the fixed point.
b. Is the system stable or unstable? Is the fixed point attracting or repelling? Why?

Example 2. Consider the DS $A_{n+2} = 5A_{n+1} - 6A_n + 2$, $n = 0, 1, 2, \ldots$. The general solution is

$$A_n = c_1 2^n + c_2 3^n + 1$$

a. Find the fixed point.
b. Is the system stable or unstable? Is the fixed point attracting or repelling? Why?

Example 3. Consider the DS $A_{n+2} = \frac{5}{2}A_{n+1} - A_n + 2$, $n = 0, 1, 2, \ldots$. The general solution is

$$A_n = c_1 \left(\frac{1}{2}\right)^n + c_2 2^n + \frac{4}{5}$$

a. Find the fixed point.
b. Is the system stable or unstable? Is the fixed point attracting or repelling? Why?

Example 4. Consider the DS $A_{n+2} = -\frac{1}{2}A_{n+1} + \frac{1}{2}A_n + 1$, $n = 0, 1, 2, \ldots$. The general solution is

$$A_n = c_1 \left(\frac{1}{2}\right)^n + c_2 (-1)^n + 1$$

a. Find the fixed point.
b. Is the system stable or unstable? Is the fixed point attracting or repelling? Why?

3 The case $a + b = 1$

- We saw earlier that in this case, the second order linear DS has either 0 or infinite fixed points
- We will only examine the long term behavior of the general solution: is the DS stable or unstable?
  - We are not concerned about whether the fixed points (if they exist) are attracting or repelling

Example 5. Consider the DS $A_{n+2} = 6A_{n+1} - 5A_n$, $n = 0, 1, 2, \ldots$. The general solution is

$$A_n = c_1 5^n + c_2$$

Is the system stable or unstable? Why?

Example 6. Consider the DS $A_{n+2} = 2A_{n+1} - A_n + 6$, $n = 0, 1, 2, \ldots$. The general solution is

$$A_n = c_1 + c_2 n + 3n^2$$

Is the system stable or unstable? Why?

Example 7. Consider the DS $A_{n+2} = A_n$, $n = 0, 1, 2, \ldots$. The general solution is

$$A_n = c_1(-1)^n + c_2$$

Is the system stable or unstable? Why?