Lesson 11. Introduction to Matrices

1 Overview

- Last time: models with six variables and six equations
- What if we have a model with hundreds of variables and equations? Thousands?
- **Matrix algebra** enables us to handle large systems of linear equations in a concise way
  - Important for equilibrium analysis (a.k.a. comparative statics), econometrics, optimization
  - Some types of nonlinear systems can be transformed into or approximated by systems of linear equations

2 What is a matrix?

- A **matrix** is a rectangular array of numbers, symbols, or expressions
- For example:

$$A = \begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad c = \begin{bmatrix} 22 & 12 & 10 \end{bmatrix}$$

- The individual items in a matrix are called its **elements** (or **entries**)
- By convention:

$$\begin{aligned} A_{ij} &= \text{the element in the } i\text{th row and } j\text{th column of matrix } A \\ &= \text{“the } ij \text{ element of } A\text{”} \end{aligned}$$

- The **dimension** (or **size**) of a matrix with m rows and n columns is $m \times n$ (“m by n”)
- “Row then column!”

Example 1.

a. What is the dimension of A? x? c?
b. What is A_{23}? A_{32}? c_{12}?

- A **row vector** is a matrix with only one row
- A **column vector** is a matrix with only one column

3 Matrix equality, addition and subtraction

- Two matrices are equal if and only if
  - they have the same dimension
  - their corresponding elements are identical
    - i.e. the ij element of one matrix is equal to the ij element of the other
- For example:

$$\begin{bmatrix} 4 & 3 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 2 & 0 \end{bmatrix} \neq \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$$

- When we add two matrices of the same dimension, we get another matrix of the same dimension
- We add two matrices by adding their corresponding elements
- We subtract two matrices by subtracting their corresponding elements

Example 2. Compute the following.

a. $\begin{bmatrix} 4 & 9 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} =$

b. $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} =$

- Note: you cannot add or subtract two matrices of different dimension!

4 Scalar multiplication

- When we multiply a matrix by a scalar (a number), we get another matrix of the same dimension
- We multiply a matrix by a scalar by multiplying each element of the matrix by the scalar

Example 3. Find the following products.

a. $7\begin{bmatrix} 3 & -1 \\ 0 & 5 \end{bmatrix} =$

b. $-1\begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{bmatrix} =$

5 Multiplying vectors: the dot product

- The **dot product** of vectors $u = [u_1, u_2, \dots, u_n]$ and $v = [v_1, v_2, \dots, v_n]$ is

- The dot product is well-defined only when u and v have the same number of elements
- u and v can be row or column vectors

6 Matrix multiplication

- How do we multiply two matrices together?
- Let A be an $m \times n$ matrix, and let B be an $n \times p$ matrix
  - Note: (# columns of A) = (# rows of B)

Example 4. Quick check: What does the ith row of A look like? What does the jth column of B look like?

- The product AB is an $m \times p$ matrix
- To get the ij element of AB:
  - We take the ith row of A and jth column of B
  - Multiply corresponding their elements and add it all up:

- In other words, the ij element of AB is the dot product of the ith row of A and the jth column of B

Example 5. Let

$$A = \begin{bmatrix} 6 & -5 & 1 \\ 1 & 0 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -1 \\ 5 & 2 \\ 0 & 1 \end{bmatrix}$$

a. What is the dimension of AB? What is the dimension of BA?
b. Find AB and BA.

- Note: order of multiplication matters! Usually, $AB \neq BA$

> **Example 6.** Give two matrices A and B such that AB is not well-defined.

- If A is an $n \times n$ matrix and k is a positive integer, then A^k is defined as:

- If $v = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$ is a $1 \times n$ row vector and $w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$ is a $n \times 1$ column vector, then:

7 Matrices act like scalars under addition

- $A - B = A + (-B)$
- **Commutative law.** For any two matrices A, B:
- **Associative law.** For any three matrices A, B, C:

8 Matrices don't always act like scalars under multiplication

- As we saw in Example 5, matrix multiplicative is not commutative: $AB \neq BA$
- Since order matters in multiplication, we have terminology that specifies the order

- In the product AB:
  - B is **premultiplied** by A
  - A is **postmultiplied** by B
- **Associative law.** For any three matrices A, B, C:

- **Distributive law.** For any three matrices A, B, C:

9 With your neighbor

Example 7. Let

$$A = \begin{bmatrix} 3 & 6 \\ 4 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} -4 & 0 \\ 2 & 5 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & -1 & 3 \\ 0 & 6 & 2 \end{bmatrix}$$

Compute AC, BC and $(B + A)C$.

Example 8. Compute the following:

a. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} =$

b. $\begin{bmatrix} 3 & -2 & 4 \\ -9 & 8 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$

Example 9. Compute the following:

a. $\begin{bmatrix} -2 & 4 & 1 \\ 8 & 6 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} =$

b. $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} =$

Example 10. Let

$$C = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad E = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$$

Find CD and CE.

10 Identity matrices

- An **identity matrix** is a square matrix with 1s in its principal diagonal (northwest to southeast) and 0s everywhere else:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- I_n is the $n \times n$ identity matrix
- What happened in Example 8?
- The identity matrix plays the role that "1" has with scalars
- For any matrix A, we have
- What is $(I_n)^2 = (I_n)(I_n)$? How about $(I_n)^k$ for any integer $k \geq 1$?

11 Null matrices

- A **null matrix** (or **zero matrix**) is a matrix whose elements are all 0
- A null matrix is not restricted to being square
  - It's important to keep track of a null matrix's dimension
- We denote a null matrix by 0:

$$\underset{(2\times 2)}{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \underset{(2\times 3)}{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- What happened in Example 9a?
- The null matrix plays the role that "0" has with scalars
- For any matrix A, we have:

12 Matrix algebra can be weird

- Unlike algebra with scalars, $AB = 0$ does not necessarily imply either $A = 0$ or $B = 0$
  - To illustrate, recall Example 9b
- Also unlike algebra with scalars, $CD = CE$ does not necessarily imply $D = E$
  - To illustrate, recall Example 10

13 The transpose of a matrix

- Let A be an $m \times n$ matrix
- The **transpose** of A is denoted by A^T
  - A^T has dimension $n \times m$
  - The columns of A are the rows of A^T
  - Similarly, the rows of A are the columns of A^T

Example 11. Let

$$A = \begin{bmatrix} 3 & 8 & -9 \\ 1 & 0 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 9 & 7 \\ 3 & 7 & 1 \end{bmatrix}$$

Find A^T and B^T.

- A matrix B is **symmetric** if $B = B^T$
  - What are some examples of symmetric matrices?
- Properties of transposes:
  - $(A^T)^T = A$
  - $(A+B)^T = A^T + B^T$
  - $(AB)^T = B^T A^T$

Example 12. Let A and B the same matrices as in Example 5 on page 3. Find $B^T A^T$.