Lesson 15. More Economic Applications of Linear Systems

1 Overview

- In this lesson/worksheet, you will solve two types of economic models using the techniques for solving systems of linear equations we covered in Lessons 12 and 13.

2 Market models

- Consider the following **two commodity market model**:

$$
\begin{aligned}
D_1 &= S_1 && (1) \qquad & D_2 &= S_2 && (4)\\
D_1 &= 10 - 2P_1 + P_2 && (2) \qquad & D_2 &= 15 + P_1 - P_2 && (5)\\
S_1 &= -2 + 3P_1 && (3) \qquad & S_2 &= -1 + 2P_2 && (6)
\end{aligned}
$$

where

$$
\begin{aligned}
D_1 &= \text{demand for product 1} \qquad & D_2 &= \text{demand for product 2}\\
S_1 &= \text{supply for product 1} \qquad & S_2 &= \text{supply for product 2}\\
P_1 &= \text{price for product 1} \qquad & P_2 &= \text{price for product 2}
\end{aligned}
$$

- Look at equations (2) and (5). Explain why product 1 and product 2 are substitutes. (See Lesson 10 for a refresher.)

$$
\left.\begin{aligned}
P_2 \uparrow &\Rightarrow D_1 \uparrow\\
P_1 \uparrow &\Rightarrow D_2 \uparrow
\end{aligned}\right\}
$$

Therefore, product 1 and product 2 are substitutes.

- We want to find the equilibrium prices P_1 and P_2 in this market.

- First, simplify the system (1)-(6) above. Because of equation (1), you can set the right hand sides of equations (2) and (3) equal to each other. You should obtain an equation with 2 variables: P_1 and P_2. Simplify the equation by collecting terms and putting all the P_1 and P_2 terms on the left, and the constant on the right.

$$10 - 2P_1 + P_2 = -2 + 3P_1$$

$$\Rightarrow \quad -5P_1 + P_2 = -12 \qquad \text{(A)}$$

- Do the same with equations (4), (5) and (6):

$$15 + P_1 - P_2 = -1 + 2P_2$$

$$\Rightarrow \quad P_1 - 3P_2 = -16 \qquad \text{(B)}$$

- Putting together the equations you found in (A) and (B), you should end up with the following system of equations:

$$\begin{aligned} -5P_1 + P_2 &= -12 \\ P_1 - 3P_2 &= -16 \end{aligned} \tag{C}$$

(Did you get the same equations? You may have the same equations, but multiplied by −1, which is OK.)

- Rewrite the system (C) in matrix form $AX = B$:

$$A = \begin{bmatrix} -5 & 1 \\ 1 & -3 \end{bmatrix} \qquad X = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \qquad B = \begin{bmatrix} -12 \\ -16 \end{bmatrix}$$

- Now, use Cramer's rule to solve this system and find the equilibrium prices:

$$P_1 = \frac{\begin{vmatrix} -12 & 1 \\ -16 & -3 \end{vmatrix}}{\begin{vmatrix} -5 & 1 \\ 1 & -3 \end{vmatrix}} = \frac{36+16}{15-1} = \frac{52}{14} = \frac{26}{7}$$

$$P_2 = \frac{\begin{vmatrix} -5 & -12 \\ 1 & -16 \end{vmatrix}}{\begin{vmatrix} -5 & 1 \\ 1 & -3 \end{vmatrix}} = \frac{80+12}{14} = \frac{92}{14} = \frac{46}{7}$$

- You should find that the equilibrium prices are $P_1 = \frac{26}{7}$ and $P_2 = \frac{46}{7}$.

3 A model for national income

- Consider the following **national income model**:

$$Y = C + I_0 + G_0 \tag{7}$$

$$C = a + bY \qquad (0 < b < 1) \tag{8}$$

where

$$\begin{aligned} Y &= \text{national income} \\ C &= \text{consumer expenditure} \\ I_0 &= \text{business expenditure (i.e., investment)} \\ G_0 &= \text{government expenditure} \end{aligned}$$

(We saw a more complicated version of this model back in Lesson 7.)

- Equation (7) says that national income equals total expenditure by consumers, business, and government.

- What does equation (8) say about the relationship between consumer expenditure and total national income?

Consumer expenditure depends directly on total national income:
as $Y \uparrow$, then $C \uparrow$

- Now suppose that $I_0 = 8$, $G_0 = 5$, $a = 4$, and $b = \frac{1}{3}$. Then equations (7) and (8) become

$$Y = C + 13 \tag{9}$$
$$C = 4 + \frac{1}{3}Y \tag{10}$$

- We want to solve for the national income Y and consumer expenditures C.
- First, simplify equations (9) and (10) by putting the Y and C terms on the left, and the constants on the right:

$$\begin{aligned} Y - C &= 13 \\ -\tfrac{1}{3}Y + C &= 4 \end{aligned} \tag{D}$$

- Rewrite the system of equations you wrote in (D) in matrix form $AX = B$:

$$A = \begin{bmatrix} 1 & -1 \\ -\frac{1}{3} & 1 \end{bmatrix} \qquad X = \begin{bmatrix} Y \\ C \end{bmatrix} \qquad B = \begin{bmatrix} 13 \\ 4 \end{bmatrix}$$

- Now, use Cramer's rule to solve this system and find the national income and consumer expenditure:

$$Y = \frac{\begin{vmatrix} 13 & -1 \\ 4 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -\frac{1}{3} & 1 \end{vmatrix}} = \frac{13+4}{1-\frac{1}{3}} = \frac{17}{2/3} = \frac{51}{2}$$

$$C = \frac{\begin{vmatrix} 1 & 13 \\ -\frac{1}{3} & 4 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -\frac{1}{3} & 1 \end{vmatrix}} = \frac{4+\frac{13}{3}}{1-\frac{1}{3}} = \frac{25/3}{2/3} = \frac{25}{2}$$

- You should find that the national income $Y = \frac{51}{2}$ and consumer expenditure $C = \frac{25}{2}$.

4 Exercises

Problem 1. Consider the three commodity market model given by

$$D_1 = S_1 \qquad D_2 = S_2 \qquad D_3 = S_3$$
$$D_1 = 2 - P_1 + 2P_2 + 2P_3 \qquad D_2 = 5 + 2P_1 - P_2 + 2P_3 \qquad D_3 = 5 - 2P_1 + 2P_2 - P_3$$
$$S_1 = -2 + P_1 \qquad S_2 = -1 + P_2 \qquad S_3 = -3 + P_3$$

Using a similar method to the one outlined in Section 2 of this worksheet, simplify the above model into a system of 3 linear equations and 3 variables P_1, P_2, and P_3. Solve this system to find the equilibrium prices P_1, P_2, and P_3 by forming the augmented matrix of the system and finding the RREF.

Problem 2. Suppose that in a national income model as in Section 3, we have $I_0 = 12$, $G_0 = 4$, $a = 1$, $b = \frac{1}{4}$. Use Cramer's rule to find Y and C.

$\boxed{1}$ Setting $D_1 = S_1$, etc.:

$$\left.\begin{aligned} 2 - P_1 + 2P_2 + 2P_3 &= -2 + P_1 \\ 5 - 2P_1 - P_2 + 2P_3 &= -1 + P_2 \\ 5 - 2P_1 + 2P_2 - P_3 &= -3 + P_3 \end{aligned}\right\} \Rightarrow \begin{aligned} -2P_1 + 2P_2 + 2P_3 &= -4 \\ -2P_1 - 2P_2 + 2P_3 &= -6 \\ -2P_1 + 2P_2 - 2P_3 &= -8 \end{aligned}$$

Augmented matrix:

$$\begin{bmatrix} -2 & 2 & 2 & -4 \\ -2 & -2 & 2 & -6 \\ -2 & 2 & -2 & -8 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_1} \begin{bmatrix} 1 & -1 & -1 & 2 \\ -2 & -2 & 2 & -6 \\ -2 & 2 & -2 & -8 \end{bmatrix} \xrightarrow[R_3 + 2R_1]{R_2 + 2R_1} \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & -4 & 0 & -2 \\ 0 & 0 & -4 & -4 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{4}R_2} \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & -4 & -4 \end{bmatrix} \xrightarrow{-\frac{1}{4}R_3} \underbrace{\begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{\text{REF}} \xrightarrow{R_1 + R_2} \begin{bmatrix} 1 & 0 & -1 & \frac{5}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 + R_3} \underbrace{\begin{bmatrix} 1 & 0 & 0 & \frac{7}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{RREF}}$$

So, $P_1 = \frac{7}{2}$, $P_2 = \frac{1}{2}$, $P_3 = 1$

$\boxed{2}$ $I_0 = 12$, $G_0 = 4$, $a = 1$, $b = \frac{1}{4}$

Model:
$$\left.\begin{aligned} Y &= C + 16 \\ C &= 1 + \tfrac{1}{4}Y \end{aligned}\right\} \Rightarrow \begin{aligned} Y - C &= 16 \\ -\tfrac{1}{4}Y + C &= 1 \end{aligned}$$

Using Cramer's rule:

$$Y = \frac{\begin{vmatrix} 16 & -1 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -\frac{1}{4} & 1 \end{vmatrix}} = \frac{16+1}{1-\frac{1}{4}} = \frac{17}{3/4} = \frac{68}{3}$$

$$C = \frac{\begin{vmatrix} 1 & 16 \\ -\frac{1}{4} & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -\frac{1}{4} & 1 \end{vmatrix}} = \frac{1+4}{1-\frac{1}{4}} = \frac{5}{3/4} = \frac{20}{3}$$