Lesson 16. Optimization of Functions with n Variables

1 Review: optimization of a function of 2 variables

- Let f be a function of two variables
- $f(a,b)$ is a **local minimum** of f if $f(a,b) \leq f(x,y)$ for all (x,y) "near" (a,b)
- $f(a,b)$ is a **local maximum** of f if $f(a,b) \geq f(x,y)$ for all (x,y) "near" (a,b)
- (a,b) is a **critical point** of f if either

  (i) $f_x(a,b) = 0$ and $f_y(a,b) = 0$ <u>or</u> (ii) $f_x(a,b)$ or $f_y(a,b)$ does not exist

- <u>Local optima must occur at critical points</u>
- How to find local optima:
  - Let's assume f_x, f_y, f_{xx}, f_{yy}, and f_{xy} always exist
  - Let (a,b) be a critical point of f – in this case, that means $f_x(a,b) = 0$ and $f_y(a,b) = 0$
  - **Second derivative test.**
    - Define $D = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$
    - Then:

if	$D > 0$	and	$f_{xx}(a,b) > 0$,	then f has a local minimum at (a,b)
if	$D > 0$	and	$f_{xx}(a,b) < 0$,	then f has a local maximum at (a,b)
if	$D < 0$,			then f has a saddle point at (a,b)
if	$D = 0$,			then the test gives no information

Example 1. Find the local optima of $f(x,y) = 12x + 18y - 2x^2 - xy - 2y^2$.

2 An economic application: profit maximization for a multiproduct firm

- There are many applications of optimization to economics
- A classic example: profit maximization
- Consider a firm that produces and sells two products
- Prices of these products are exogenously determined
- Variables:

$$R = \text{revenue} \qquad Q_1 = \text{quantity of product 1 produced}$$
$$C = \text{cost} \qquad Q_2 = \text{quantity of product 2 produced}$$

- Model:

$$\begin{aligned} \text{maximize} \quad & R - C \\ \text{subject to} \quad & R = 12Q_1 + 18Q_2 \\ & C = 2Q_1^2 + Q_1Q_2 + 2Q_2^2 \end{aligned}$$

- The unit price of product 1 is [ ] and the unit price of product 2 is [ ]
- The marginal cost of product 1 is [ ]
- The marginal cost of product 2 is [ ]
- The production costs of the two products are related to each other!
- We can write profit as a function of Q_1 and Q_2:

[ ]

- We want to maximize profit π – we already did this in Example 1!

$$f \leftrightarrow \pi \qquad x \leftrightarrow Q_1 \qquad y \leftrightarrow Q_2$$

- Locally optimal production plan and profit:

[ ]

- Looking ahead: what if we have 3 products? 100 products? n products?

3 The gradient and critical points

- Let f be a function of n variables
- Let's call these variables $x_1, x_2, \ldots, x_n$
- $f(a_1, a_2, \ldots, a_n)$ is a **local minimum** of f if

$$f(a_1, a_2, \ldots, a_n) \leq f(x_1, x_2, \ldots, x_n) \quad \text{for all } (x_1, x_2, \ldots, x_n) \text{ “near” } (a_1, a_2, \ldots, a_n)$$

- $f(a_1, a_2, \ldots, a_n)$ is a **local maximum** of f if

$$f(a_1, a_2, \ldots, a_n) \geq f(x_1, x_2, \ldots, x_n) \quad \text{for all } (x_1, x_2, \ldots, x_n) \text{ “near” } (a_1, a_2, \ldots, a_n)$$

- Let's assume that all the first and second partial derivatives always exist
- The **gradient** of f is the vector

- In words, $\frac{\partial f}{\partial x_i}(a_1, a_2, \ldots, a_n)$ is

- Intuitively, the rate of change at a local minimum or local maximum should be zero in all directions
- **Theorem.** If $(a_1, a_2, \ldots, a_n)$ is a local minimum or local maximum of f, then $\nabla f(a_1, a_2, \ldots, a_n) = 0$, or equivalently

$$\frac{\partial f}{\partial x_1}(a_1, \ldots, a_n) = 0 \qquad \frac{\partial f}{\partial x_2}(a_1, \ldots, a_n) = 0 \qquad \cdots \qquad \frac{\partial f}{\partial x_n}(a_1, \ldots, a_n) = 0$$

- The points that satisfy the first-order necessary condition are called **critical points**
- Note that this is just a more general version of what we had for functions with 2 variables

Example 2. Find the critical points of $f(x_1, x_2, x_3) = e^{2x_1} + e^{-x_2} + e^{x_3^2} - 2x_1 - 2e^{x_3} + x_2$.

4 The Hessian and the second derivative test

- How do we know if a critical point is a local minimum or a local maximum?
- We need a "second derivative test" for n variables
- The **Hessian matrix** of f is

$$H(x_1, \ldots, x_n) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

- Recall that $\frac{\partial^2 f}{\partial x_j \partial x_i}$ means "take the derivative of f with respect to x_i, then with respect to x_j"

Example 3. Find the Hessian matrix of $f(x_1, x_2, x_3) = e^{2x_1} + e^{-x_2} + e^{x_3^2} - 2x_1 - 2e^{x_3} + x_2$. Recall from Example 2 that

$$\frac{\partial f}{\partial x_1} = 2e^{2x_1} - 2 \qquad \frac{\partial f}{\partial x_2} = -e^{-x_2} + 1 \qquad \frac{\partial f}{\partial x_3} = 2x_3 e^{x_3^2} - 2e^{x_3}$$

- Let $H_k(x_1, \ldots, x_n)$ be the square <u>sub</u>matrix formed by the first k rows and columns of $H(x_1, \ldots, x_n)$
- The k**th principal minor** of $H(x_1, \ldots, x_n)$ is

Example 4. Find all of the leading principal minors of $H(x_1, \ldots, x_n)$ from Example 3 at $(x_1, x_2, x_3) = (0, 0, 1)$, the critical point of f found in Example 2.

- **Second derivative test.**
  - Suppose $(a_1, a_2, \ldots, a_n)$ is a critical point of f
  - If $d_n \neq 0$:

    (1) if all the principal minors of $H(a_1, \ldots, a_n)$ are positive $(d_1 > 0, d_2 > 0, \ldots, d_n > 0)$ — then f has a local minimum at $(a_1, \ldots, a_n)$

    (2) if the first principal minor of $H(a_1, \ldots, a_n)$ is negative and the remaining principal minors alternate in sign $(d_1 < 0, d_2 > 0, d_3 < 0, \text{etc.})$ — then f has a local maximum at $(a_1, \ldots, a_n)$

    (3) otherwise, — f has a saddle point at $(a_1, \ldots, a_n)$
  - If $d_n = 0$, then the test gives no information

Example 5. Is $(x_1, x_2, x_3) = (0, 0, 1)$, the critical point of f found in Example 2, a local minimum or a local maximum?

- Note that the second-order sufficient condition is just a more general version of the second derivative test we had for functions with 2 variables
- For a function $f(x, y)$ with 2 variables, the Hessian is

  and so

(i) "all the principal minors are positive" means

(ii) "the first principal minor is negative, and the remaining principal minors alternate in sign" means

5 Exercises

Problem 1. Let $f(x, y, z) = x^3 + xy^2 + x^2 + y^2 + 3z^2$. Find the critical points. Classify each critical point as a local maximum, local minimum, or saddle point.