$$H(6,9,5) = \begin{bmatrix} -8 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & -12 \end{bmatrix}$$

- The principal minors of the Hessian at $(Q_1, Q_2, Q_3) = (6, 9, 5)$ are

$|H_1| = -8 \qquad |H_2| = 80 \qquad |H_3| = -960$

- Therefore, the second derivative test tells us that

π has a local maximum at $(6, 9, 5)$

- So, the company's locally optimal production plan and profit is:

$Q_1 = 6,\ Q_2 = 9,\ Q_3 = 5 \qquad \pi(6, 9, 5) = 679$

3 Exercises

Problem 1. Supppose we have a company that manufactures two products that are sold in the same market. The company has a monopoly and may charge whatever prices it wishes. Let

R = revenue	Q_1 = quantity of product 1 produced and sold	P_1 = unit price of product 1
C = cost	Q_2 = quantity of product 2 produced and sold	P_2 = unit price of product 2

Assume that the demand of the two products depends on their prices as follows:

$$\left.\begin{aligned} Q_1 &= 40 - 2P_1 + P_2 \\ Q_2 &= 15 + P_1 - P_2 \end{aligned}\right\} (*)$$

In addition, assume the cost of production is $C = Q_1^2 + Q_1Q_2 + Q_2^2$. How much of each product should the company manufacture in order to maximize total profit?

Use demand curves to solve for P_1, P_2 in terms of Q_1, Q_2:

$$(*): \left.\begin{aligned} -2P_1 + P_2 &= Q_1 - 40 \\ P_1 - P_2 &= Q_2 - 15 \end{aligned}\right\} \Rightarrow \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} Q_1 - 40 \\ Q_2 - 15 \end{bmatrix}$$

Cramer's rule:

$$P_1 = \frac{\begin{vmatrix} Q_1 - 40 & 1 \\ Q_2 - 15 & -1 \end{vmatrix}}{\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix}} = 55 - Q_1 - Q_2 \qquad P_2 = \frac{\begin{vmatrix} -2 & Q_1 - 40 \\ 1 & Q_2 - 15 \end{vmatrix}}{\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix}} = 70 - Q_1 - 2Q_2$$

$$\Rightarrow \pi(Q_1, Q_2) = P_1Q_1 + P_2Q_2 - C$$

$$= (55 - Q_1 - Q_2)Q_1 + (70 - Q_1 - 2Q_2)Q_2 - Q_1^2 - Q_1Q_2 - Q_2^2$$

$$= 55Q_1 - Q_1^2 - Q_1Q_2 + 70Q_2 - Q_1Q_2 - 2Q_2^2 - Q_1^2 - Q_1Q_2 - Q_2^2$$

$$= 55Q_1 - 2Q_1^2 + 70Q_2 - 3Q_2^2 - 3Q_1Q_2$$

$$\frac{\partial \pi}{\partial Q_1} = 55 - 4Q_1 - 3Q_2 \qquad \frac{\partial \pi}{\partial Q_2} = 70 - 6Q_2 - 3Q_1$$

CP: $\left.\begin{aligned} 55 - 4Q_1 - 3Q_2 &= 0 \\ 70 - 3Q_1 - 6Q_2 &= 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} 4Q_1 + 3Q_2 &= 55 \\ 3Q_1 + 6Q_2 &= 70 \end{aligned}\right\} \Rightarrow (Q_1, Q_2) = (8, \tfrac{23}{3})$

By Cramer's rule, substitution, or calculator

$$H(Q_1, Q_2) = \begin{bmatrix} -4 & -3 \\ -3 & -6 \end{bmatrix}$$

2nd deriv test: $H(8, \tfrac{23}{3}) = \begin{bmatrix} -4 & -3 \\ -3 & -6 \end{bmatrix} \qquad |H_1| = -4 \qquad |H_2| = 15$

$\Rightarrow (8, \tfrac{23}{3})$ is a local max

Locally optimal production plan: $Q_1 = 8$, $Q_2 = \tfrac{23}{3}$

Locally optimal profit: $\pi(8, \tfrac{23}{3}) \approx 488.33$